Identity-Compatible Auctions*

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Abstract

This paper studies the incentives of the seller and buyers to shill bid in a single-item auction. An auction is seller identity-compatible if the seller cannot profit from pretending to be one or more bidders via fake identities. It is buyer identity-compatible if no buyer profits from posing as more than one bidder. Lit auctions reveal the number of bidders, whereas dark auctions conceal the information. We characterize three classic selling mechanisms—first-price, second-price, and posted-price—based on identity compatibility. We show the importance of concealing the number of bidders, which enables the implementation of a broader range of outcome rules. In particular, no optimal lit auction is ex-post seller identity-compatible, while the dark first-price auction (with reserve) achieves the goal.

Keywords: Mechanism Design, Auction, Identity Compatibility, Shill Bidding, Lit, Dark
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1 Introduction

Shill bidding can be challenging to identify and prevent in practice. Historically, even when physical attendance at auctions was required because of technical constraints, multiple bidders might still have acted on behalf of a buyer or the seller. These bidders often used subtle, hard-to-detect signals to communicate their intentions. "Such signals may be in the form of a wink, a nod, scratching an ear, lifting a pencil, tugging the coat of the auctioneer, or even staring into the auctioneer's eyes—all of them perfectly legal" (Cassady, 1967).

In today's marketplaces, it becomes increasingly hard to detect shill bidding. As reported by the Wall Street Journal,[1] most bids come in online or by telephone in major auction houses like Christie's or Sotheby's, which makes it difficult to verify the identities of bidders. A long-standing practice in the industry is "chandelier bidding," under which auctioneers declare a series of fictitious bids to drum up excitement within the crowd. It used to be only legal until reaching the secret reserve price, i.e., the minimum price at which a consignor was willing to sell. Recently, in an unexpected move, New York City repealed such regulation, which made many experts fear that it would erode people's trust and confidence.[2] Unsurprisingly, internet auctions are particularly susceptible to shill bidding, because buyers and sellers do not reveal their identities and trade goods anonymously. Despite the risk of legal consequences,[3] shill bidding remains pervasive due to its ease of implementation and the significant challenges involved in detecting it. For example, Chen, Liang, Chang, Liu, Yin and Yu (2020) estimate that in eBay Motors auctions, approximately 9% of bidders are shill bidders and around 22% of all listings contain shill bids. While shill bidding is typically associated with the seller, buyers can also exploit it profitably. Seibel and Škoda (2023) document that buyers can employ shills to crowd out competitors during the preselection stage in two-stage auctions (Example 2).

Shill bidding raises concerns not only for buyers and the seller but also for the marketplace as a whole. Despite the deregulation in New York City, major auction houses have expressed their intent to operate as if the previous regulations were still in place (Herrera, 2023), even though they lack full commitment to enforcing them (Akbarpour and Li, 2020). eBay emphasizes that "we want to maintain a fair marketplace for all our users, and as such, shill bidding is prohibited on eBay.... eBay has a number of systems in place to detect and monitor bidding patterns and practices. If we identify any malicious behavior, we'll take steps to

[1] Why auction rooms seem empty these days. The Wall Street Journal, June 15, 2014.

[2] New York City Eliminates the Rules That Govern Art and Other Auctions, The New York Times, May 3, 2022.

[3] 3 Men Are Charged With Fraud In 1,100 Art Auctions on EBay, The New York Times, March 9, 2001.

prevent it."[4] But so far, those measures appear to be only partially effective.[5] Marketplaces aim to build a reputation for fairness, yet they often struggle with challenges due to a lack of commitment power and credibility. For instance, Google, as the operator of the largest online displaying advertising market, is facing a lawsuit over alleged shill-bidding-like manipulations of ad auctions.[6][7]

Given the importance of preventing shill bidding, which auction designs are most effective in deterring this practice? This paper answers this question by examining which auction formats are incentive-compatible for buyers and/or the seller to truthfully report their *identities*—that is, who they truly are. We differentiate between *bidders*—identities showing up in the auction—and *buyers*, who act on behalf of their own sincere interests for purchase. Shill bidding occurs when the seller uses fake identities to pretend to be one or more bidders, or when a single buyer adopts more than one identity to pose as multiple bidders. We assume that no one knows who controls which bidders (identities) except for their own.

The challenge of identifying shill bidding stems from the uncertainty regarding the number of buyers. In the absence of such uncertainty, shill bidding becomes evident when the number of bidders participating in the auction exceeds the number of buyers. This stands in contrast with the standard auction theory, which assumes common knowledge of a fixed number of buyers (Vickrey, 1961; Myerson, 1981; Milgrom and Weber, 1982). It follows that when the designer proposes a mechanism, the exact number of buyers is not yet known.[8] Accordingly, the proposed mechanism must be capable of accommodating varying numbers of buyers, unlike standard auction models where the number of buyers is considered a fixed primitive.[9] Hence, we model the auction game in the presence of shill bidding as follows.

The game begins with the designer publicly announcing the mechanism. Following this, both buyers and the seller simultaneously decide how many identities to employ, which

[4]Shill Bidding Policy, eBay.

[5]There are still many recent discussions about shill bidding on eBay posted online.

[6]Google Misled Publishers and Advertisers, Unredacted Lawsuit Alleges, The Wall Street Journal, Jan 14, 2022.

[7]Google Broke the Law to Keep Its Advertising Monopoly, a Judge Rules, The New York Times, April 17, 2025.

[8]One key consideration in practical auction design is attracting buyers (Bulow and Klemperer, 1996; Klemperer, 2002; Milgrom, 2004, 2017), as increased competition—driven by a higher number of buyers—can significantly raise the auction's revenue (Holt, 1979; McAfee and McMillan, 1987a). The uncertainty regarding the number of buyers is a typical feature of many real-world auctions, even in the absence of shill bidding.

[9]Consider the first-price auction as an example. In practice, the number of buyers is often uncertain, so the first-price auction can be viewed as a collection of standard first-price auctions, each corresponding to a different finite number of buyers. The auction effectively operates as a collection of these scenarios.

together determine the total number of bidders. Finally, the specific auction corresponding to this total number of bidders is conducted, with both buyers and the seller controlling the bidders (identities) they have employed. Notably, the designer observes only the bidders (identities) rather than the underlying seller or buyers who control them, and the auction format may vary depending on the number of bidders. For example, the designer might run the first-price auction when there are no more than five bidders, and switch to the second-price auction otherwise. Shill bidding offers two channels for participants to manipulate the auction: first, by altering the auction format through the selection of a specific number of bidders, and second, by optimizing the strategies of the bidders they control within the specific auction. I study the implications of shill bidding in the symmetric independent private values model (Myerson, 1981). Symmetry arises from anonymity because the mechanism cannot discriminate against bidders (identities), which are ex-ante identical.

A new design element emerges when modeling the mechanism as a collection of standard auctions: whether the number of bidders is common knowledge or not. When the standard auction theory assumes common knowledge of a fixed number of buyers, it implicitly encompasses two assumptions: first, that the number of bidders is common knowledge, and second, that each bidder represents a distinct buyer. In the presence of shill bidding, the second assumption no longer holds true, and the first assumption transforms into a design choice.[10] I will slightly abuse the terminology and refer to the mechanism as an auction, even though it actually consists of standard auctions, each corresponding to a different finite number of bidders. An auction is a *lit* auction if the number of bidders is common knowledge. An auction is a *dark* auction if the number of bidders is concealed. I do not differentiate between auctions that induce the same equilibrium outcome for each type profile. In this paper, a collection of first-price (or Dutch) auctions, each corresponding to a different finite number of bidders, is called the *first-price* auction; similarly, a collection of second-price (or English) auctions is called the *second-price* auction; a collection of posted-price mechanisms with the same price is called the *posted-price* mechanism.[11]

To deter shill bidding, the auction should ensure that the seller finds it unprofitable to participate via multiple identities, and buyers find it unprofitable to control more than one identity. In other words, it is incentive-compatible for them to report their true identities rather than exploiting fake ones. We refer to this property as *identity compatibility*, and explore different notions of it as follows.[12] Bayesian (buyer or seller) identity compatibility

[10]Notice that the number of bidders does not necessarily equal the number of buyers because of shill bidding.

[11]In the posted-price mechanism, the seller proposes a take-it-or-leave-it price to buyers. If multiple buyers accept the price, the seller breaks the tie among them.

[12]Incentive compatibility "requires that no one should find it profitable to "cheat," where cheating

entails no profitability in expectation (for buyers or the seller respectively). Ex-post (buyer or seller) identity compatibility is a stronger notion, ensuring no profitability for every type profile of others. Finally, ex-post *auctioneer* identity compatibility further strengthens ex-post seller identity compatibility by accounting for the possibility that the seller and the auctioneer collude to deviate from the auction rules in ways that remain undetectable to buyers, even when buyers can share information after the auction.[13]

The first main result (Theorem 1) shows that the second-price auction with (or without) the optimal reserve price is the unique optimal (or optimally efficient) auction that is ex-post buyer identity-compatible. This follows from the fact that ex-post buyer identity compatibility implies strategy-proofness (Lemma 1). It holds in both lit and dark auctions, because strategy-proofness renders the number of bidders irrelevant. Furthermore, we extend this result to the finite type space (Theorem 6). It is important to note, however, that the standard characterization of the second-price auction under strategy-proofness does not directly generalize to the finite type space.[14] Hence, this result provides a novel characterization of the second-price auction that is valid across both finite and continuous type spaces.

The second main result (Theorem 2) is an impossibility theorem: no optimal (or optimally efficient) lit auction is ex-post seller identity-compatible. This result underscores a fundamental dilemma for the seller. On one hand, competition in the auction is essential to maximize revenue. On the other hand, the seller cannot credibly commit to refraining from exploiting the competition for their benefit through the use of identities. Consequently, it is impossible to simultaneously achieve both optimality and ex-post seller identity compatibility in lit auctions, reflecting the fundamental tension between revenue maximization and the creation of fake competition by the seller. For any optimal lit auction, the seller always has an incentive to participate via identities to increase the revenue.

Given this tension, I proceed to identify the best auction for the seller—one that frees them from any suspicion of manipulating the auction in their favor while maximizing revenue. In particular, I take into account the possibility that the seller colludes with the auctioneer. It turns out that the posted-price mechanism generates the highest expected revenue among all lit auctions that are ex-post auctioneer identity-compatible (Theorem 3). It reinforces the previous result by showing that the dilemma not only constrains the seller but does so in

is defined as behavior that can be made to look "legal" by a misrepresentation of a participant's preferences or endowment" (Hurwicz, 1972). Identity compatibility requires that no one should find it profitable to "cheat" by deceiving others using fake identities.

[13]In some cases, the seller is also the auctioneer.

[14]See Holmström (1979), Harris and Raviv (1981), Lovejoy (2006), Elkind (2007), and Jeong and Pycia (2023).

the strictest way that they have to completely tie their hands. The rigidity of the payment is the cost the seller has to incur in order to maintain self-discipline.

Subsequently, I turn to dark auctions due to the impossibility result for lit auctions. I demonstrate that the dark first-price auction with (or without) the optimal reserve price is the unique optimal (or optimally efficient) dark auction that is ex-post auctioneer identity-compatible and Bayesian buyer identity-compatible (Theorem 4). In other words, we see "light in the dark." The intuition behind this is that the pay-as-bid property ensures that the seller cannot manipulate the payment even if they know the type profile of buyers. Moreover, since the number of bidders is concealed in dark auctions, the seller is unable to heighten the perceived competition among buyers by employing multiple identities. The dilemma highlighted in Theorem 2 is addressed by converting the competition essential for achieving optimality into a fixed bid through the concealment of the number of bidders. The key thing to notice here is that competition in the auction can be analyzed from two perspectives: intensive competition, which pertains to a fixed number of bidders, and extensive competition, which arises from variations in the number of bidders. The intensive competition for a fixed number of bidders can be integrated into a fixed bid, as in the first-price auction, ensuring it cannot be manipulated by other participants. The problem for lit auctions is that the extensive competition can always be exploited by the seller, e.g., the equilibrium bid in the first-price auction increases with the number of bidders. The dark first-price auction resolves this problem by incorporating the extensive competition into a fixed bid across different numbers of bidders, thereby preventing such exploitation.

Together, these results imply an auction dilemma: an optimal auction cannot simultaneously achieve ex-post identity compatibility for both buyers and the seller. This dilemma can be resolved by relaxing the requirement from "ex-post" to "Bayesian" for either party.

Finally, I discuss how these results extend to dynamic auctions and partitional disclosure policies. By examining identity compatibility in auctions, this paper characterizes the first-price auction, the second-price auction, and the posted-price mechanism. The comparison between lit and dark auctions underscores the significance of concealing the number of bidders in the presence of shill bidding, as it allows for the implementation of a broader range of outcome rules (Theorem 5).

It is well known in practice that information disclosure in auctions can facilitate collusion by providing a mechanism for signalling and punishment, highlighting that transparency is not inherently beneficial (Cramton and Schwartz, 2000; Klemperer, 2002, 2003).[15] This paper reinforces that point from a straightforward perspective, showing that when the

[15] In Hong Kong 3G auction, transparency was abused by strong bidders to lobby for a "small" change of design, which made entry harder and collusion easier in the end. Interestingly, they ridiculed the original design by calling it the "dark auction." (See Footnote 48 in Klemperer (2003).)

disclosed information—specifically, the number of bidders—can be falsified by fake identities to mislead participants, it is better to conceal it.

1.1 Related Literature

I am far from the first to consider shill bidding in auctions. The previous literature focuses on shill bidding separately by the seller and buyers. Since my analysis addresses both, I first review the literature on buyers and then on the seller.

Sakurai, Yokoo and Matsubara (1999) and Yokoo, Sakurai and Matsubara (2004) refer to shill bids from buyers as *false name* bids. They focus on *false-name-proof* combinatorial auctions and emphasize that the failure of the substitutes condition makes the VCG[16] auction vulnerable to false name bids, which is in the same spirit as the collusion problem examined by Ausubel and Milgrom (2002; 2005). Sher (2012) studies the optimal shill bidding strategies in the VCG auction for heterogenous objects. For the single-unit setting, Arnosti, Beck and Milgrom (2016) combine false-name-proofness with adverse selection in advertising auctions. Different from mine, they directly focus on strategy-proof auctions.

There is a richer literature on shill bidding by the seller. Most of them restrict attention to the case that the seller uses only one shill and study how the seller makes best use of it. Graham, Marshall and Richard (1990) study the English auction with heterogenous buyers, and show that the seller may benefit from shill bidding in expectation by using a non-constant reserve price when buyers are indistinguishable. Relatedly, Izmalkov (2004) shows that the seller cannot do so in the English auction with personalized optimal reserve prices when buyers are distinguishable.[17] Chakraborty and Kosmopoulou (2004) consider the English auction with common values, and show that the seller can get worse off in expectation by the possibility of shill bidding if buyers take that into account when bidding. With interdependent values, Lamy (2009) shows that the linkage effect (Milgrom and Weber, 1982) is reduced by the shill-bidding effect that buyers fear that they are bidding against the seller in the second-price auction. Levin and Peck (2023) characterize the optimal shill bidding strategy for the seller in the English auction with common values. Compared to the previous literature, this paper shows that both how the seller places bids through a shill and how many shills the seller uses in the auction matter. In particular, the seller can manipulate the perceived competition among buyers by using multiple shills, which has received little attention in the literature.

In a parallel work, Shinozaki (2024) characterizes the posted-price rule as the unique

[16] The Vickrey-Clarke-Groves mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973).

[17] Lamy (2013) studies how shill bidding affects the second-price auction with costly entry.

shill-proof one when focusing on deterministic and strategy-proof mechanisms.[18] Theorem 3 in this paper allows for stochastic mechanisms and does not rely on strategy-proofness. In a contemporary work, Komo, Kominers and Roughgarden (2024) establish the Dutch auction (with reserve) as the unique optimal public auction that is *strongly shill-proof*, where they fix the number of bidders and exclude bids of zero from being classified as shill bids.[19] In contrast, I consider the presence of shills in the auction to be shill bidding, regardless of the bids placed through them. My impossibility result (Theorem 2) is not restricted to the environment with a fixed number of bidders. Consequently, I argue the importance of concealing the number of bidders in the presence of shill bidding (Theorem 4).[20]

Identity compatibility is related to the concept of *credibility* introduced by Akbarpour and Li (2020), but they are inherently different. Akbarpour and Li (2020) focus on the private communication between buyers and the auctioneer, and the deviation of the auctioneer from auction rules. In contrast, shill bidding itself does not involve deviations from the rules. With shills, the seller can only communicate "publicly" with buyers by participating in the auction as one or more bidders, whereas the number of bidders is fixed in Akbarpour and Li (2020). In particular, credibility does not consider the deviation of the auctioneer by misreporting the number of buyers in the auction. As a result, the first-price auction is credible but not Bayesian seller identity-compatible, because the seller can boost the perceived competition by "making the auction room crowded."

The paper also contributes to the literature on characterizations of auction formats. For the first-price (or Dutch) auction, see Akbarpour and Li (2020), Pycia and Raghavan (2021), Jeong and Pycia (2023), Komo, Kominers and Roughgarden (2024), and Häfner, Pycia and Zeng (2024). For the second-price (or English) auction, see Green and Laffont (1977), Holmström (1979), Li (2017), Akbarpour and Li (2020), and Pycia and Troyan (2023). For the posted-price mechanism, see Hagerty and Rogerson (1987), Čopič and Ponsatí (2016), Andreyanov, Čopič and Jeong (2018), Pycia and Troyan (2023), and Shinozaki (2024). Previous results work with various concepts, like strategy-proofness, robustness, credibility, simplicity, non-bossiness, and leakage-proofness. This paper unifies all characterization results by varying different notions of identity compatibility. Finally, the way of modeling the mechanism as a collection of standard auctions connects to the literature on uncertain number of buyers in auctions (McAfee and McMillan, 1987b; Matthews, 1987; Harstad,

[18] Shill-proofness in Shinozaki (2024) is similar to ex-post auctioneer identity compatibility (Definition 14) in this paper.

[19] Strong shill-proofness is similar to ex-post auctioneer identity compatibility, but the seller has to dynamically adjust the number of shills to keep the number of bidders fixed when different numbers of buyers realize, no matter what the seller's bidding strategies are.

[20] Wang, Hidvégi, Zoltán and Whinston (2004) study how to deter the seller from shill bidding by charging a commission fee.

Kagel and Levin, 1990; Lauermann and Speit, 2023).

1.2 Outline of the Paper

In Section 2, we present the model and incorporate identities into auctions. Section 3 defines identity compatibility and explores its implications in lit auctions. In Section 4, we introduce dark auctions and revisit the concept of identity compatibility. Section 5 concludes with further extensions. Omitted proofs can be found in Appendix A and Online Appendix B.

2 Model

We index all potential agents with natural numbers $\mathbb{N} = \{1, 2, \dots\}$. Agents are ex-ante identical and indistinguishable.[21] Agents' types are denoted by $\theta_i \in \Theta_i = \Theta$ for all $i \in \mathbb{N}$, which are independent and identically distributed.[22] Let $\mathcal{N}$ denote the collection of all finite subsets of the set of potential agents $\mathbb{N}$.[23] Let $\mathcal{B}$ be a random variable representing the set of agents, with realizations denoted by $B \in \mathcal{N}$. Then, $\sum_{B\in\mathcal{N}} \mathbb{P}(\mathcal{B} = B) = 1$. For any finite set of agents $N \in \mathcal{N}$, there is a set of outcomes X^N, with generic element $x^N = \left(x_i^N\right)_{i\in N}$.[24] Each agent $i \in N$ has a von Neumann-Morgenstern utility function $u_i : X^N \times \Theta_i \to \mathbb{R}$.[25]

2.1 Collection of Games

When the set of agents is random, the designer proposes a collection of games $\Gamma^{\mathcal{N}} = \left(\Gamma^N\right)_{N\in\mathcal{N}}$, where each game Γ^N is designed for each finite set of agents $N \in \mathcal{N}$. Consider the game Γ^N. Let M_i^N denote the set of messages for agent $i \in N$, with generic element m_i^N. A strategy S_i^N for agent i maps each type θ_i to a message m_i^N, i.e., $S_i^N : \Theta_i \to M_i^N$.[26] Let Σ_i^N denote the set of strategies for agent i in the game Γ^N. Let $M^N = \times_{i\in N} M_i^N$ and $g^{\Gamma^N} = \left(g_i^{\Gamma^N}\right)_{i\in N} : M^N \to X^N$ be a function that maps each message profile $m^N =$

[21] In Section 5.4, we consider the case in which agents are distinguishable and their types are not necessarily identically distributed.

[22] Formally, for each agent $i \in \mathbb{N}$, the type space Θ_i is endowed with σ-algebra $\mathcal{F}_i$. There is a probability measure $D_i : \mathcal{F}_i \to [0, 1]$. The distribution is identical across agents, i.e., $D_i = D_j$ for all $i, j \in \mathbb{N}$.

[23] $\mathcal{N}$ is countable, because the set of all finite subsets of a countable set is countable.

[24] We allow for stochastic outcomes.

[25] When agents are indistinguishable, the utility function u_i may vary with the number of agents $|N|$. In contrast, when agents are distinguishable, the utility function should be denoted by u_i^N, indicating that it may vary with the specific identities of the set of agents N.

[26] Mixed strategies are allowed when m_i^N is treated as a generic mixed message.

$\left(m_i^N\right)_{i\in N} \in M^N$ to an outcome $x^N = \left(x_i^N\right)_{i\in N} \in X^N$. When agents submit a message profile m^N, agent i obtains the outcome $g_i^{\Gamma^N}\left(m^N\right)$.

Because agents are indistinguishable, the designer is unable to discriminate against agents' identities. Hence, the collection of games $\Gamma^{\mathcal{N}}$ is *anonymous* in the sense that it treats any group of agents of the same size identically and the outcome is independent of their individual identities. Let $|N|$ denote the cardinality of the set of agents N.

Definition 1. $\Gamma^{\mathcal{N}} = \left(\Gamma^N\right)_{N\in\mathcal{N}}$ is *anonymous* if, for all $N, N' \in \mathcal{N}$ such that $|N| = |N'|$, all bijective functions $\phi : N \to N'$, and all $m^N \in M^N$, we have $g_i^{\Gamma^N}\left(m^N\right) = g_{\phi(i)}^{\Gamma^{N'}}\left(\hat{m}^{N'}\right)$, where $m_i^N = \hat{m}_{\phi(i)}^{N'}$ for all $i \in N$.

In other words, Γ^N and $\Gamma^{N'}$ are identical up to a relabeling of agents for all $N, N' \in \mathcal{N}$ such that $|N| = |N'|$. As a result, we can rewrite $\Gamma^{\mathcal{N}}$ as a collection of symmetric games $\Gamma^{\mathbb{N}} = (\Gamma^n)_{n\in\mathbb{N}}$. For any set of agents N with size $|N| = n$, the agents play the game Γ^n. Symmetry follows directly from anonymity, as agents are treated identically in the game. The strategy space Σ^n for each agent in the game Γ^n satisfies $\Sigma^n = \Sigma_i^N = \Sigma_j^{N'}$ for all $i \in N$ and $j \in N'$, where $|N| = |N'| = n$. We focus on symmetric equilibria.[27] Let S^n denote the type-strategy for each agent in the game Γ^n that maps from types to strategies, i.e., $S^n : \Theta \to \Sigma^n$. When agents play according to $S^n(\theta_N) = (S^n(\theta_i))_{i\in N}$, the resulting outcome is $g^{\Gamma^n}(S^n(\theta_N))$. Let $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right) = (\Gamma^n, S^n)_{n\in\mathbb{N}}$ denote the collection of games and strategy profiles.

Definition 2. $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right)$ is *lit Bayesian incentive-compatible* if, for all $n \in \mathbb{N}$, all $i \in \mathbb{N}$, and all $\theta_i \in \Theta$,

$$S^n(\theta_i) \in \arg\max_{\sigma^n\in\Sigma^n} \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(\sigma^n, S^n(\theta_{-i})\right), \theta_i\right)\right].$$

We call $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right)$ a *lit mechanism* if it is lit Bayesian incentive-compatible. The term "lit" highlights the implicit assumption that each agent is aware of the number of agents participating in the game. As this notion is standard in the literature, we will occasionally omit the qualifier and refer to lit mechanisms simply as mechanisms when the context is clear. To avoid confusion early on, we postpone the definition of *dark* mechanisms (Definition 15) to Section 4.1. We further discuss lit and dark mechanisms in environments where agents

[27] Restricting attention to symmetric equilibria follows the usual practice in the literature, given the symmetric environment under consideration. Moreover, this restriction entails no loss of generality with respect to the implementation of an anonymous outcome rule—i.e., one that depends solely on the type profile of agents and not on their identities. Specifically, for any asymmetric equilibrium in the game Γ^n that implements such a rule, one can construct a symmetric equilibrium in an augmented game by introducing an initial randomization stage. In this stage, agents are randomly assigned to roles in the asymmetric strategy profile, thereby inducing a symmetric equilibrium that preserves the implementation of the anonymous outcome rule.

are distinguishable in Section 5.4.

We define an equivalence relation between mechanisms. Two mechanisms $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right)$ and $\left(\hat{\Gamma}^{\mathbb{N}}, \hat{S}^{\mathbb{N}}\right)$ are *equivalent* if, for all $n \in \mathbb{N}$ and all type profiles $\theta_N \in \Theta^n$, the distribution over outcomes from the strategy profile $S^n(\theta_N)$ in the game Γ^n is the same as that from the strategy profile $\hat{S}^n(\theta_N)$ in the game $\hat{\Gamma}^n$.

Definition 3. Two mechanisms $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right)$ and $\left(\hat{\Gamma}^{\mathbb{N}}, \hat{S}^{\mathbb{N}}\right)$ are *equivalent* if, for all $n \in \mathbb{N}$, we have $g^{\Gamma^n}(S^n(\cdot)) = g^{\hat{\Gamma}_n}\left(\hat{S}^n(\cdot)\right)$.

This equivalence definition is purely outcome-based. Equivalent mechanisms implement the same outcome rule, i.e., the same mapping from type profiles to outcomes. In this paper, we do not differentiate between equivalent mechanisms and focus on direct mechanisms. As shown in Section 5.3, this restriction entails no loss of generality. All of our results extend to indirect (or dynamic) mechanisms as well.

2.2 Auctions with an Unknown Number of Buyers

2.2.1 Identities

There is only one item for sale. Accordingly, the random variable $\mathcal{B}$ denotes the set of buyers attracted to the item, with $B \in \mathcal{N}$ representing a realized set of buyers. We assume that the expected number of buyers is finite, i.e., $\sum_{n\in\mathbb{N}} n\mathbb{P}(|\mathcal{B}| = n) < \infty$.[28] We assume that $\mathbb{P}(|\mathcal{B}| = 1) > 0$.[29] Each buyer's type θ_i corresponds to i's valuation for the item to be sold. Assume that $\Theta_i = \Theta = [0,1]$ and that θ_i is independently identically distributed according to a continuous full-support density $f : [0,1] \to \mathbb{R}$. The cumulative distribution function is denoted by $F(\theta_i) = \int_0^{\theta_i} f(x)\,dx$. The seller is denoted by 0.

Both the seller and buyers can act under possibly multiple (fake) identities. Let N_i denote the set of identities employed by buyer $i \in B$. In particular, each buyer's true identity is included in this set, i.e., $i \in N_i$. The collection of all identities used by buyers is denoted by $N_B = \cup_{i\in B} N_i$. When buyers do not shill bid, we have $B = N_B$. Let S denote the set of identities employed by the seller. When the seller does not shill bid, we have $S = \emptyset$. The complete set of identities is then given by $N = N_B \cup S$.[30]

[28] The distribution of $\mathcal{B}$ does not matter for lit auctions. For dark auctions, we assume that buyers have common priors (Definition 16).

[29] This assumption implies that whenever there is more than one bidder in the auction, there is a risk of shill bidding. Conversely, there does not exist a nontrivial number of bidders for which the absence of shill bidding can be guaranteed.

[30] For any two distinct buyers $i \neq j$, the sets of identities controlled by them are disjoint, i.e., $N_i \cap N_j = \emptyset$. Besides, the sets of identities controlled by buyers and the seller are disjoint, i.e., $N_B \cap S = \emptyset$.

Importantly, all identities are indistinguishable from each other. Hence, the designer cannot discriminate ex-ante between identities used by buyers and the seller, which necessitates the design of an anonymous mechanism $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right)$. Each identity is treated as a bidder (agent). Given the set of identities N, buyers and the seller play the game $\Gamma^{|N|}$. For simplicity, we will refer to a mechanism as an *auction*, and use the terms "identity" and "bidder" interchangeably.

2.2.2 Auctions

For any fixed set of bidders $N = N_B \cup S$, the outcome $x^N = (q^N, t^N)$ consists of the allocation $q^N = \left(q_i^N\right)_{i \in N}$ and the payment $t^N = \left(t_i^N\right)_{i \in N}$, where $\sum_{i \in N} q_i^N \leq 1$, $q_i^N \in [0,1]$, and $t_i^N \in \mathbb{R}$. The seller's value for the item is normalized to zero. They desire revenue. The utility function is

$$u_0 = \sum_{i \in N_B} t_i^N,$$

which sums up the payments from all bidders (identities) controlled by buyers. Each buyer $i \in B$ has quasi-linear utilities,

$$u_i = \left(\sum_{j \in N_i} q_j^N\right) \theta_i - \left(\sum_{j \in N_i} t_j^N\right).$$

The probability of buyer i obtaining the item is equal to the sum of the winning probabilities of all identities controlled by buyer i. Similarly, the total payment for buyer i is the sum of the payments associated with those identities.

An (anonymous) auction $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right)$ induces an allocation rule $q^{(\Gamma^n, S^n)}(\cdot)$ and a payment rule $t^{(\Gamma^n, S^n)}(\cdot)$ for all $n \in \mathbb{N}$. For any set of bidders $N \in \mathcal{N}$ with $|N| = n$, the outcome induced by the auction is given by

$$\left(q^{(\Gamma^n, S^n)}(\theta_N), t^{(\Gamma^n, S^n)}(\theta_N)\right) = x^{(\Gamma^n, S^n)}(\theta_N) = g^{\Gamma^n}\left(S^n(\theta_N)\right) \quad \forall \theta_N \in \Theta^n.$$

We denote the collection of induced outcome rules $\left(q^{(\Gamma^n, S^n)}, t^{(\Gamma^n, S^n)}\right)_{n \in \mathbb{N}}$ simply by (q, t) to ease the notation. Specifically, for all $n \in \mathbb{N}$, and all $N \in \mathcal{N}$ with $|N| = n$, we have $q(\theta_N) = q^{(\Gamma^n, S^n)}(\theta_N)$ and $t(\theta_N) = t^{(\Gamma^n, S^n)}(\theta_N)$ for all $\theta_N \in \Theta^n$. This simplification is justified by the anonymity of the auction, which ensures that the outcome depends solely on the type profile and the number of bidders, and not on their identities.

Throughout the remainder of the text, we will use (q, t) to refer to the auction and will not distinguish between auctions that are equivalent (Definition 3). In other words, we focus on direct auctions. For example, we reinterpret the classic selling mechanisms in our setting as follows.[31]

1. The *first-price* auction: Run the first-price (or Dutch) auction for all $N \in \mathcal{N}$.

2. The *second-price* auction: Run the second-price (or English) auction for all $N \in \mathcal{N}$.

3. The *posted-price* mechanism: Run the same posted-price mechanism for all $N \in \mathcal{N}$.[32]

Our definition of an auction differs from the standard one, because it is actually a collection of standard auctions, each corresponding to a different finite number of bidders. For example, the first-price auction in our setting is actually a collection of first-price auctions in the standard setting, each corresponding to a different finite number of bidders $|N|$. In general, we allow the outcome rule to vary with $|N|$. For instance, we can run the first-price auction when there are no more than five bidders, and switch to the second-price auction otherwise. We also allow for auctions with a cap on the number of bidders, by explicitly taking into account the preselection stage (Example 2).[33]

Now that the auction has been defined, we can outline the complete game, taking into account shill bidding.

1. The auctioneer publicly announces an auction (q, t).

2. A set of buyers $B \in \mathcal{N}$ is drawn, which is not revealed to any buyer or the seller.

3. Buyers learn their types privately and independently.

4. Each buyer $i \in B$ and the seller simultaneously decide how many identities to employ in the auction, i.e., N_i and S.

5. A standard auction starts with a fixed set of bidders $N = N_B \cup S = (\cup_{i \in B} N_i) \cup S$.

In practice, steps 4 and 5 can occur simultaneously if the auction allows entry after it begins. If such an auction effectively deters the seller or buyers from shill bidding, it will continue to do so even if entry is prohibited after the auction starts. All the results presented in

[31] The first-price auction and the Dutch auction induce the same outcome rule, as do the second-price auction and the English auction.

[32] The posted-price mechanism is a take-it-or-leave-it offer. Ties are broken among bidders who accept the posted price.

[33] In auctions that impose a cap on the number of participants, a preselection stage becomes necessary whenever the number of interested buyers exceeds the specified threshold. This is consistent with our framework, where for any $N \in \mathcal{N}$, the designer initiates the preselection stage, if necessary, prior to conducting the main auction. An extensive-form game can explicitly integrate the preselection stage into the auction's game structure. Such dynamic auctions are considered in this paper, though we restrict attention to their induced outcome rules. We show in Section 5.3 that this restriction is without loss, and provide further details on the implementation via extensive forms.

this paper remain valid under conditions where entry is allowed after the auction begins. Therefore, the distinction between steps 4 and 5 can be made without any loss of generality.

At the final step, we do not specify whether participants know how many bidders are in the auction. The number of bidders in the auction is assumed common knowledge in the standard auction theory (Vickrey, 1961; Myerson, 1981; Milgrom and Weber, 1982). In our setting, whether the designer discloses this information becomes a design choice. In principle, bidders can participate in the auction without the knowledge of the number of bidders, because the auction format is publicly announced for any given number of bidders. In fact, this is how many real-world auctions operate. At auction houses and online auction platforms, no one really knows the exact number of bidders, and the same auction format is simply fixed for all $N \in \mathcal{N}$.[34]

Definition 4. A *lit* auction is an auction where the number of bidders $|N|$ is common knowledge. A *dark* auction is an auction where the number of bidders $|N|$ is concealed.

Throughout the paper, we do not consider how buyers update their beliefs about the number of buyers or whether the seller shill bids in the auction. This is justified by our focus on auction formats that discourage shill bidding. In such auctions, shill bidding does not arise in equilibrium, and buyers rationally believe every other bidder is a distinct buyer. Because shill bidding is undetectable under anonymity, when we examine the incentives for shill bidding in auctions designed to deter such behavior, buyers should maintain the belief that all other bidders are distinct buyers.[35] Consequently, in lit auctions, where the number of bidders is common knowledge, buyers take it for granted that the number of buyers equals the number of bidders.[36] In dark auctions, where the number of bidders is concealed, they act under the belief that the unobserved number of bidders always equals the number of buyers.

In the following, we focus on auctions that are *ex-post individually rational*.

Definition 5. An auction is *ex-post individually rational* if, for all $N \in \mathcal{N}$, all $i \in N$, and all

[34] At auction houses like Christie's or Sotheby's, due to the decline in physical attendance, bidders do not know exactly how many others are participating online or by phone (Footnote 1). In eBay auctions, bidders do not know how many "snipers" are watching the auction closely behind the computer's screen (Roth and Ockenfels, 2002). In Google's advertising auctions, advertisers are unaware of how many others are competing in the auction.

[35] In contrast, when analyzing auctions that do not effectively deter shill bidding—which lies outside the scope of this paper—it would be inappropriate to assume that buyers believe all other bidders are distinct buyers.

[36] For lit auctions, if $\mathbb{P}(|\mathcal{B}| = k) = 0$ for some $k \in \mathbb{N}$, then buyers do not interpret the presence of k bidders as evidence of shill bidding. Those are zero-probability events and are treated as off-path information sets under weak perfect Bayesian equilibrium. In such cases, we assume buyers still hold the belief that every other bidder is a distinct buyer, which is the most favorable scenario for shill bidding.

$\theta_N \in \Theta^{|N|}$,

$$\theta_i q_i(\theta_N) - t_i(\theta_N) \geq 0.$$

The revenue of the auction is the sum of the payments from all buyers.

Definition 6. An auction is *optimal* if it maximizes expected revenue subject to ex-post individual rationality.

A lit auction is optimal if it is optimal for each $N \in \mathcal{N}$, whereas an optimal dark auction does not entail it is optimal for each $N \in \mathcal{N}$, due to the concealment of the number of bidders (Section 4.2). Myerson (1981) characterizes the optimal (lit) auction's allocation rule in terms of the virtual valuation $v(\theta_i) = \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$.

Definition 7. The type distribution is *regular*, if the virtual valuation $v(\cdot)$ is strictly increasing.

We assume the type distribution is regular. The optimal reserve price ρ^* is determined by

$$\rho^* = \min\{\theta_i \in [0,1] | v(\theta_i) \geq 0\}.$$

Note that efficiency[37] does not determine the payoffs of buyers of the lowest type. To eliminate subsidies from the seller,[38] we focus on efficient auctions that maximize expected revenue, which is referred to as *optimal efficiency*.[39]

Definition 8. An auction is *optimally efficient* if it maximizes expected revenue subject to ex-post individual rationality and efficiency.

In the following section, we explore identity compatibility in lit auctions, where the number of bidders is common knowledge. When the context is clear, we simply refer to a lit auction as an auction. We will turn to dark auctions in Section 4.

3 Identity Compatibility in Lit Auctions

Identity compatibility requires that the seller should have no incentive to participate in the auction under the disguise of one or more bidders, and buyers should find it optimal to use only their true identities, implying $B = N_B = N$. To conduct a detailed analysis, we begin by examining the incentives of buyers and the seller separately. Subsequently, in Section 5.1, we consider identity compatibility for both parties jointly. Throughout, when referring to

[37] In this paper, an efficient auction is defined as a collection of standard efficient auctions for each $N \in \mathcal{N}$.

[38] When subsidies are available, an impractical way to deter shill bidding from the seller is to have them provide higher subsidies as the number of bidders increases.

[39] We do not directly assume that buyers of the lowest type obtain zero payoffs.

an auction as (q,t), we implicitly restrict attention to direct mechanisms. As discussed in Section 5.3, this restriction entails no loss of generality.

3.1 Buyer Identity Compatibility

In this subsection, we first define the concept of Bayesian buyer identity compatibility and illustrate how shill bidding distorts the auction format while expanding the strategy space for buyers. We then introduce a natural strengthening—ex-post buyer identity compatibility—which leads to our first main result: a characterization of the second-price auction in terms of buyer identity compatibility.

Definition 9. An auction (q,t) is *Bayesian buyer identity-compatible* if, for all $i \in \mathbb{N}$, all $\theta_i \in \Theta$, all $|N_i| \in \mathbb{N}$, and all $\hat{\theta}^1_{N_i}, \hat{\theta}^2_{N_i}, \cdots \in \Theta^{|N_i|}$,

$$\begin{aligned}
&\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\theta_i q_i\left(\theta_i,\theta_{-i}\right)-t_i\left(\theta_i,\theta_{-i}\right)\right]\middle| i\in B\right]\\
\geq&\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\sum_{j\in N_i}\left[\theta_i q_j\left(\hat{\theta}^{|B|}_{N_i},\theta_{-i}\right)-t_j\left(\hat{\theta}^{|B|}_{N_i},\theta_{-i}\right)\right]\right]\middle| i\in B\right].
\end{aligned}$$

Bayesian buyer identity compatibility ensures that no buyer $i \in B$ can gain an advantage by unilaterally using multiple identities $j \in N_i$ in expectation. The use of iterated expectations arises because, at the time buyer i decides how many identities to employ in the auction, i does not yet know the total number of buyers. In others words, buyer i must commit to a specific number of identities for each realization of the set of buyers. Given this commitment, their bidding strategies can still depend on the number of buyers.[40]

Notice that when $|N_i| = 1$, i.e., $N_i = \{i\}$, the above inequality is guaranteed by lit Bayesian incentive compatibility. However, when $|N_i| > 1$, buyer i controls multiple identities, creating opportunities for auction manipulation through shill bidding. This manipulation operates through two channels. First, for each realized set of buyers B, every buyer $j \in B\setminus\{i\}$ believes there are $|N|$ buyers in the auction, while only buyer i knows the actual number of buyers in the auction is $|B|$, which is less than $|N|$. Consequently, the auction format is distorted: instead of conducting the specific auction corresponding to $|B|$ buyers, the game follows the one corresponding to $|N|$ buyers. Second, the strategy space for buyer i is expanded by leveraging multiple identities. Buyer i can formulate bidding strategies represented by the type profile $\hat{\theta}^{|B|}_{N_i} \in \Theta^{|N_i|}$. The identities controlled by buyer

[40]If the auction allows entry after it begins, then buyer i is not required to commit in advance. For such auctions, Bayesian buyer identity compatibility ensures that the above inequality holds for each realized set of buyers B, even after removing the outer expectation.

i can collude, effectively acting as a bidding ring or cartel.[41] Truthful reporting of types is guaranteed under lit Bayesian incentive compatibility in the absence of shill bidding. However, in the presence of shill bidding, there are double deviations, where buyers misreport both their types and their identities. As a result, it may no longer be optimal for buyer i to bid truthfully, i.e., $\hat{\theta}_j = \theta_i$ for all $j \in N_i$.

In the following two examples, we will illustrate how shill bidding impacts the auction through the two channels discussed above. The first example focuses on how it distorts the auction format, while the second highlights its role in expanding the strategy space.

Example 1. Consider the first-price auction. It is optimal for buyer i to bid as if $\hat{\theta}_j = \hat{\theta}_{j'}$ for all $j, j' \in N_i$, since submitting a losing bid provides no advantage. The only potential benefit of using multiple identities comes from the increased chance of winning in case of a tie. However, in the continuous type space, the probability of a tie is zero. Thus, expanding the strategy space through multiple identities offers no advantage in the first-price auction. Moreover, this manipulation carries a cost: it distorts the auction format. Buyers other than i perceive a more competitive standard first-price auction with a greater number of bidders.[42] The presence of additional bidders (identities) drives up competition, leading other buyers to bid higher. As a result, securing a win becomes more expensive for buyer i. Hence, buyers have no incentive to use multiple identities in the first-price auction, as it would only increase their winning payment without improving their chances of winning.

Fact 1. *The first-price auction is Bayesian buyer identity-compatible.*

Example 2. Consider the following two-stage auction analyzed in Seibel and Škoda (2023).[43] In the preselection stage, bidders submit sealed bids, and a pre-announced, limited number of bidders with the highest bids advance. In the main stage, an English auction takes place, with the highest bid from the preselection stage serving as the starting price.[44]

Notice that the auction format is fixed in the main stage as an English auction with a fixed number of slots. As a result, using multiple identities cannot distort the auction format in the main stage. Instead, the advantage of employing multiple identities lies in the

[41] The analogy is made intuitively, as cartel members differ from identities. A buyer has full control over identities, whereas cartel members may cheat on each other. Moreover, cartels consist of distinct buyers, while identities are merely duplicates of the same buyer. For the literature on collusion in auctions, see Robinson (1985), Graham and Marshall (1987), Mailath and Zemsky (1991), McAfee and McMillan (1992), and Chassang and Ortner (2019).

[42] We consider standard first-price auctions with different finite numbers of bidders as distinct standard auctions.

[43] This is a dynamic auction. As discussed in Section 5.3, all results hold under dynamic auctions, although the focus is primarily on the induced outcome rule.

[44] This is a Dutch-Anglo auction—not an Anglo-Dutch auction (Klemperer, 1998).

expanded strategy space. The identities (bidders) controlled by a firm essentially form a cartel. This cartel can submit tying high bids to try to occupy all slots in the main stage, thereby kicking out competitors.[45] Only rival firms who outbid the cartel can then proceed to the main stage. If the cartel successfully crowds out all competitors from participating in the main stage, the English auction will conclude at the reserve price, with all but one bidder dropping out immediately.[46] This allows the firm controlling the cartel to win at a low cost. Using multiple identities is profitable because the exclusion of competitors forestalls price competition in the main stage.

Seibel and Škoda (2023) develop these theoretical predictions, and empirically validate them by using administrative data on public procurement auctions in Slovakia, along with a court-confirmed cartel case. Their findings demonstrate that this two-stage auction fails Bayesian buyer identity compatibility.[47] In general, multistage auctions that incorporate a preselection stage may be susceptible to shill bidding by buyers, as they can use multiple identities to crowd out competitors.

A natural strengthening of Bayesian buyer identity compatibility is the following concept:

Definition 10. An auction (q, t) is *ex-post buyer identity-compatible* if, for all $i \in \mathbb{N}$, all $\theta_i \in \Theta$, and all $|N_i| \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\theta_i q_i\left(\theta_i,\theta_{-i}\right)-t_i\left(\theta_i,\theta_{-i}\right)\right]\Big|\, i\in B\right]\\ \geq &\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\sup_{\hat{\theta}_{N_i}\in\Theta^{|N_i|}}\sum_{j\in N_i}\left[\theta_i q_j\left(\hat{\theta}_{N_i},\theta_{-i}\right)-t_j\left(\hat{\theta}_{N_i},\theta_{-i}\right)\right]\right]\Bigg|\, i\in B\right].\end{aligned}$$

Ex-post buyer identity compatibility ensures that no buyer $i \in B$ can gain an advantage by unilaterally using multiple identities $j \in N_i$ in any case, although i has to commit to the number of identities in advance. When $|N_i| = 1$, i.e., $N_i = \{i\}$, the above inequality reduces to

$$\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\theta_i q_i\left(\theta_i,\theta_{-i}\right)-t_i\left(\theta_i,\theta_{-i}\right)\right]\Big|\, i\in B\right]$$

[45] Competitors will place higher bids in the preselection stage due to the presence of more bidders (identities). However, this effect is limited since the preselection stage influences the winning payment only by setting the starting price for the main stage.

[46] The buyer's immediate withdrawal during the main stage represents an off-path deviation, which is not profitable when using a single identity, as such a withdrawal results in losing the auction. However, by employing multiple identities simultaneously, the buyer can crowd out competitors and secure the auction by withdrawing all but one identity at once. See Section 5.3 for further discussion of off-path deviations.

[47] Conversely, when an auction is not Bayesian buyer identity-compatible, concerns about collusion arise.

$$\geq \mathbb{E}_B \left[\mathbb{E}_{\theta_{-i} \in \Theta^{|B|-1}} \left[\sup_{\hat{\theta}_i \in \Theta} \left[\theta_i q_i \left(\hat{\theta}_i, \theta_{-i} \right) - t_i \left(\hat{\theta}_i, \theta_{-i} \right) \right] \right] \middle| i \in B \right].$$

Notice that for all $|B| \in \mathbb{N}$ and all $\theta_{-i} \in \Theta^{|B|-1}$, we always have

$$\sup_{\hat{\theta}_i \in \Theta} \left[\theta_i q_i \left(\hat{\theta}_i, \theta_{-i} \right) - t_i \left(\hat{\theta}_i, \theta_{-i} \right) \right] \geq \theta_i q_i \left(\theta_i, \theta_{-i} \right) - t_i \left(\theta_i, \theta_{-i} \right).$$

Hence, to achieve ex-post buyer identity compatibility when $|N_i| = 1$, we must have

$$\theta_i q_i \left(\theta_i, \theta_{-i} \right) - t_i \left(\theta_i, \theta_{-i} \right) = \sup_{\hat{\theta}_i \in \Theta} \left[\theta_i q_i \left(\hat{\theta}_i, \theta_{-i} \right) - t_i \left(\hat{\theta}_i, \theta_{-i} \right) \right],$$

which implies strategy-proofness.[48]

Lemma 1. *Ex-post buyer identity compatibility implies strategy-proofness.*

The converse is generally not true.[49] For instance, while the posted-price mechanism is strategy-proof, it is not even Bayesian buyer identity-compatible. This is because buyers can always exploit multiple identities to increase the chance of winning in case of a tie.[50] However, such benefit does not emerge in the second-price auction, because buyers obtain zero payoffs in case of a tie. In fact, the second-price auction is not only ex-post buyer identity-compatible but also the unique auction that simultaneously guarantees optimality (or optimal efficiency).[51]

Theorem 1. *The second-price auction with the reserve price ρ^* is the unique optimal auction that is ex-post buyer identity-compatible. The second-price auction is the unique optimally efficient auction that is ex-post buyer identity-compatible.*

The proof is relegated to Appendix A.1. Theorem 1 follows directly from Lemma 1. In the continuous type space, the second-price auction is characterized by strategy-proofness and (optimal) efficiency, as established by Green and Laffont (1977) and Holmström (1979).

[48]In this paper, a strategy-proof auction is defined as a collection of standard strategy-proof auctions for each $B \in \mathcal{N}$. For direct auctions, strategy-proofness is equivalent to ex-post incentive compatibility.

[49]As noted in Footnote 41, using multiple identities is different from forming a cartel. Likewise, group-strategy-proofness and ex-post buyer identity compatibility are conceptually distinct, with neither being inherently stronger than the other. For instance, while the posted-price mechanism is group-strategy-proof, it is not ex-post buyer identity-compatible. Conversely, as we will see next, the second-price auction is ex-post buyer identity-compatible, but it fails to be group-strategy-proof.

[50]As a direct consequence of anonymity, the symmetric tie-breaking rule is applied.

[51]Since we do not differentiate between equivalent auctions (Definition 3) and study the continuous type space, uniqueness is defined up to outcome equivalence and a set of measure zero.

However, this characterization under strategy-proofness does not directly extend to the finite type space. In Section 5.3, we extend Theorem 1 to the finite type space (Theorem 6), demonstrating that the characterization under ex-post buyer identity compatibility is novel, as it applies to both finite and continuous type spaces.

3.2 Seller Identity Compatibility

Now, we apply the analytic approach used for buyers in the previous subsection to examine the seller's incentives for shill bidding.

Definition 11. An auction (q,t) is *Bayesian seller identity-compatible* if, for all $|S| \in \mathbb{N}$, and all $\theta_S^1, \theta_S^2, \cdots \in \Theta^{|S|}$,

$$\mathbb{E}_B\left[\mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sum_{i \in B} t_i\left(\theta_B\right)\right]\right] \geq \mathbb{E}_B\left[\mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sum_{i \in B} t_i\left(\theta_B, \theta_S^{|B|}\right)\right]\right].$$

Bayesian seller identity compatibility ensures that the seller cannot raise expected revenue by committing to a set of identities S. Unlike buyers, who attempt to lower their payments by using multiple identities, the seller aims to push up the payment of the winning buyer by exploiting multiple identities. In particular, the seller avoids winning the auction, as doing so results in no sale. Given the focus on ex-post individually rational auctions, a safe strategy for the seller is to bid as if their type is zero. Therefore, any Bayesian seller identity-compatible auction must pass the following *bidding-zero* test.

Definition 12. An auction (q,t) passes the *bidding-zero* test if, for all $|S| \in \mathbb{N}$,

$$\mathbb{E}_B\left[\mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sum_{i \in B} t_i\left(\theta_B\right)\right]\right] \geq \mathbb{E}_B\left[\mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sum_{i \in B} t_i\left(\theta_B, \underbrace{0, \ldots, 0}_{|S| \text{ times}}\right)\right]\right].$$

The bidding-zero test provides a quick method to determine that an auction is not Bayesian seller identity-compatible if it fails the test. For example, the first-price auction is not Bayesian seller identity-compatible because the seller can artificially inflate the number of bidders in the auction, thus manipulating the perceived competition among buyers—under the assumption that they believe each bidder is a distinct buyer.[52] All else being equal, the more bidders in the auction, the higher the winning payments for buyers of any type.

[52]Since the first-price auction is not Bayesian seller identity-compatible, it would be inappropriate to assume that buyers believe each bidder is a distinct buyer. However, when evaluating whether the first-price auction satisfies Bayesian seller identity compatibility, we examine the incentives for shill bidding under the assumption that buyers believe each bidder is a distinct buyer. The reasoning follows the same structure as a proof by contradiction.

Although related, Bayesian seller identity compatibility differs from the concept of credibility introduced by Akbarpour and Li (2020). In their paper, the auctioneer can manipulate the auction history subject to the constraint that it aligns with the auction format. Crucially, the auctioneer can misrepresent the preferences of buyers, but the auction format is fixed in terms of the number of buyers. Hence, the auctioneer cannot misrepresent the number of buyers by introducing shill bidders.

In contrast, Bayesian seller identity compatibility assumes that the auctioneer commits to a collection of auction formats that may vary with the number of bidders. We conceptualize the auctioneer as a third party with reputational concerns, such as auction houses or online platforms. The seller, who is distinct from the auctioneer, can only influence the winning payment by participating in the auction. The seller does not have access to the same information the auctioneer receives from bidders. In other words, the seller's available strategies and information sets are no different from any other bidder's. Although the seller cannot misrepresent the preferences of buyers, the seller can distort the auction format by employing multiple identities. Such manipulation is not considered in Akbarpour and Li (2020). As a result, we have the following fact:

Fact 2. *The first-price auction is credible but not Bayesian seller identity-compatible.*

Consider the second-price auction. The type the seller pretends to be effectively acts as a reserve price, since buyers with lower types cannot win.[53] However, when the reserve price has already been optimally set at ρ^*, the seller finds it unprofitable to participate, as noted by Izmalkov (2004).[54]

Proposition 1 (Izmalkov (2004))**.** *The second-price auction with the reserve price ρ^* is Bayesian seller identity-compatible.*

However, the second-price auction can be easily manipulated by the seller, if the seller possesses more than just distributional knowledge of buyers' types. To address this, we propose the following stronger notion of seller identity compatibility.

Definition 13. An auction (q, t) is *ex-post seller identity-compatible* if, for all $|S| \in \mathbb{N}$,

$$\mathbb{E}_B \left[\mathbb{E}_{\theta_B \in \Theta^{|B|}} \left[\sum_{i \in B} t_i \left(\theta_B\right) \right] \right] \geq \mathbb{E}_B \left[\mathbb{E}_{\theta_B \in \Theta^{|B|}} \left[\sup_{\theta_S \in \Theta^{|S|}} \sum_{i \in B} t_i \left(\theta_B, \theta_S\right) \right] \right].$$

[53] In the continuous type space, there is no risk of winning the tie for the seller when participating in the auction. In Online Appendix B.1, we show that a similar result holds in the finite type space (Proposition 5).

[54] Not every optimal auction is Bayesian seller identity-compatible. For instance, the first-price auction with the reserve price ρ^* is optimal but not Bayesian seller identity-compatible, since it fails the bidding-zero test.

Ex-post seller identity compatibility ensures that even if the seller observes the buyers' type profile, the seller still cannot raise expected revenue when committing to a set of identities. In other words, it is strategy-proof for the seller not to participate in the auction by pretending to be multiple bidders. While we do not consider it realistic to assume that the seller has full knowledge of the buyers' type profile, this concept broadly captures the idea that any violation of ex-post seller identity compatibility implies the existence of some information that, if available to the seller, would create an incentive to manipulate the auction by shill bidding. Notably, this information does not need to be perfect.

The posted-price mechanism is ex-post seller identity-compatible because the winning payment is fixed. However, it suffers from both efficiency and revenue losses. In contrast, the second-price auction (with reserve) is optimal, but the seller can always increase the winner's payment when there is no tie, i.e., when there is a gap between the highest and second-highest bids.

In general, the seller faces a dilemma. On one hand, competition among bidders is crucial for maximizing revenue. On the other hand, the seller has an incentive to exploit this competition—by leveraging identities—to further increase the winning payment. In lit auctions, optimality requires that the expected revenue is maximized for any given number of buyers. This inherent conflict of interest ultimately leads to an impossibility result.

Theorem 2. *No optimal lit auction is ex-post seller identity-compatible. No optimally efficient auction is ex-post seller identity-compatible.*

Note that we impose no restrictions on the auction format. The optimal lit auction can be implemented in any static or dynamic format.[55] Therefore, Theorem 2 presents a broad impossibility result, highlighting the conflict between revenue maximization and the creation of fake competition through identities.

We provide a brief outline of the proof, which relies on two key observations. First, we identify the first-price auction (with reserve) as the optimal lit auction that poses the greatest challenge for the seller to manipulate via identities. Given that the seller shill bids in the first-price auction by committing to a set of identities, the payment conditional on winning for buyers of each type is fixed, eliminating further opportunities for manipulation. Second, in any optimal lit auction, the expected payment for buyers of each type is determined for any given number of buyers. Importantly, while the expected payment decreases as the number of buyers increases—due to the lower probability of winning—the expected payment conditional on winning actually increases. When the seller participates in the auction, they never win. Consequently, the same buyer who would have won in the seller's absence still wins, but ends

[55] See Section 5.3 for a detailed discussion on implementation via extensive forms.

up paying more. This is because the expected payment conditional on winning increases with additional bidders, even if the auction format might have changed due to the seller's participation. As a result, the seller always profits from shill bidding in optimal lit auctions. The same logic extends to efficient auctions that maximize expected revenue—optimal efficient auctions—by removing the reserve price, leading to a corresponding impossibility result. The full proof of Theorem 2 is provided in Appendix A.2.

Our impossibility result contrasts with the findings of Komo, Kominers and Roughgarden (2024), who establish the Dutch auction as the unique optimal public auction that prevents shill bidding by the seller.[56] This difference stems from their distinct definition of shill bidding—where bids of zero are not considered shill bids—and their assumption of a fixed number of bidders. Consequently, their model shuts down the channel that shill bidding can influence the auction format by increasing the number of bidders. In particular, the seller cannot intensify the perceived competition among buyers by leveraging more identities (bidders). As we noted earlier, the first-price (or Dutch) auction is not Bayesian seller identity-compatible, as it fails the bidding-zero test.

Given the impossibility result, we aim to identify the best auction for the seller—one that not only eliminates any suspicion of shill bidding but also maximizes expected revenue. In particular, we take into account the possibility that the auctioneer colludes with the seller, which further strengthens ex-post seller identity compatibility.

Definition 14. An auction (q, t) is *ex-post auctioneer identity-compatible* if, for all $B \in \mathcal{N}$ and all $|S| \in \mathbb{N} \cup \{0\}$,

$$\mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sum_{i \in B} t_i(\theta_B)\right] \geq \mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sup_{\substack{i \in B, \theta_S \in \Theta^{|S|} \\ q_i(\theta_B, \theta_S) > 0}} \frac{t_i(\theta_B, \theta_S)}{q_i(\theta_B, \theta_S)}\right].$$

Compared to ex-post seller identity compatibility, there are two key differences. First, we no longer take the outer expectation, meaning the seller does not need to commit to the number of identities in advance. Instead, the auctioneer may either secretly disclose the number of buyers to the seller or, in some cases, the seller and auctioneer may be the same entity. Second, when a random allocation is involved, the auctioneer's randomization device cannot be trusted.[57] In practice, verifying whether randomization is conducted

[56]They derive this result in the finite type space. In Online Appendix B.4, we show that Theorem 2 holds in the finite type space. In addition to what have discussed in this paragraph, our approach differs from theirs in how we handle ties, which occur with positive probability in the finite type space. For further details, see Section B.1.

[57]This concern also motivates the concept of ex post deterministic implementation in Dworczak (2020).

properly is often challenging. If the allocation rule requires that the item remains unsold with some probability, the auctioneer can always choose to sell it to ensure payment from the buyer. Moreover, when a tie-breaking decision is needed, the auctioneer can always favor the buyer who offers the highest payment conditional on winning. These deviations remain undetectable to buyers, even if they share information after the auction.[58] In other words, ex-post auctioneer identity compatibility accounts for the safest deviations from auction rules when the seller can collude with the auctioneer.

The posted-price mechanism is ex-post auctioneer identity-compatible, but it incurs losses in both efficiency and revenue. As the next theorem shows, these losses are inevitable in lit auctions if the seller aims to eliminate any suspicion of shill bidding.

Theorem 3. *The posted-price mechanism maximizes expected revenue among all lit auctions that are ex-post auctioneer identity-compatible.*

Theorem 3 demonstrates that concerns over the seller or auctioneer steering the auction in their favor completely tie their hands. This result is striking because it does not rely on any distributional assumption about the number of buyers.[59] Compared to Shinozaki (2024), we do not restrict attention to deterministic auctions and do not impose strategy-proofness. While the exact posted price varies with the details, a posted-price mechanism always achieves the maximum expected revenue in lit auctions, subject to the constraint of ex-post auctioneer identity compatibility. The proof hinges on the observation that the payment made by any winning buyer, given any type profile of the other buyers, is bounded by the payment that same buyer would make if they were the only one. (See Appendix A.3 for details.) The intuition is that if competition among bidders drives up revenue, the seller is tempted to simulate this competition using fake identities. The elimination of this temptation leads to the rigidity of the payment. Theorem 3 provides an additional explanation for the widespread use of posted-price mechanisms in practice, beyond their simplicity.[60]

4 Dark Auctions

Theorem 3 highlights the revenue loss the seller must incur to maintain ex-post auctioneer identity compatibility. Meanwhile, the impossibility result in Theorem 2 suggests that we should look beyond lit auctions for optimal auctions that are ex-post seller identity-compatible. As discussed in the model section, bidders can participate in the auction

[58] It is similar to group-credibility in Akbarpour and Li (2020).
[59] We only need $\mathbb{P}(|\mathcal{B}| = 1) > 0$.
[60] Many sellers on eBay choose the posted-price mechanism over the English auction.

without knowing the exact number of bidders, because the specific auction format is publicly announced for any given number of bidders. In practice, many auctions operate this way. For instance, auction houses and online platforms typically do not disclose the number of interested bidders. Furthermore, with the rise of remote participation, many auctions no longer require bidders to be physically present, making it difficult for the auctioneer to know the exact number of bidders (Footnote 1 and 34).

In the following, we begin by introducing the concept of dark mechanisms and examining their relationship to lit mechanisms, demonstrating that dark mechanisms allow for the implementation of a broader range of outcome rules. We then explore the implications of this broader range for the design of optimal dark auctions. Finally, we revisit the notion of identity compatibility in dark auctions and illustrate how the transition to dark auctions overcomes the impossibility result present in lit auctions.

4.1 Dark Mechanisms

We follow the notation in Section 2.1. Consider a collection of games $\Gamma^{\mathcal{N}} = \left(\Gamma^N\right)_{N\in\mathcal{N}}$, which can be simplified as $\Gamma^{\mathbb{N}} = (\Gamma^n)_{n\in\mathbb{N}}$ under anonymity. When the number of agents is concealed, agents are allowed to play contingent strategies, i.e., a collection of strategies that depend on the number of agents. Recall that Σ^n denotes the set of strategies for each agent in the game Γ^n. When allowing for contingent strategies, we denote the strategy space for each agent by Σ^d, which is a subset of $\times_{n\in\mathbb{N}}\Sigma^n$. Importantly, Σ^d does not have to be identical to $\times_{n\in\mathbb{N}}\Sigma^n$. In fact, the design of dark mechanisms involves not only the design of the collection of games $\Gamma^{\mathbb{N}}$, but also the design of the strategy space Σ^d. As in lit mechanisms (see Footnote 27), we focus on symmetric equilibria. Let $S^d = (S^n)_{n\in\mathbb{N}}$ denote the type-strategy for each agent in the collection of games $\Gamma^{\mathbb{N}}$ that maps from types to strategies, i.e., $S^d : \Theta \to \Sigma^d$.

Definition 15. $\left(\Gamma^{\mathbb{N}}, \Sigma^d, S^d\right)$ is *dark Bayesian incentive-compatible* if, for all $i \in \mathbb{N}$ and all $\theta_i \in \Theta$,

$$S^d(\theta_i) \in \arg\max_{\sigma^d\in\Sigma^d} \sum_{n\in\mathbb{N}} p_i(n)\, \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(\sigma^n, S^n(\theta_{-i})\right), \theta_i\right)\right],$$

where $p_i(n) = \mathbb{P}\left(|\mathcal{B}| = n \mid i \in \mathcal{B}\right)$ is agent i's belief that the number of agents is n.

We call $\left(\Gamma^{\mathbb{N}}, \Sigma^d, S^d\right)$ a *dark mechanism* if it is dark Bayesian incentive-compatible. The term "dark" highlights the assumption that the number of agents participating in the game is concealed. Our first observation establishes the connection between dark and lit mechanisms.

Proposition 2. *Every lit mechanism induces an equivalent dark mechanism.*

Proof. Consider any lit mechanism $\left(\Gamma^{\mathbb{N}}, S^{\mathbb{N}}\right)$. Lit Bayesian incentive compatibility (Definition 2) implies that for all $n \in \mathbb{N}$, all $i \in \mathbb{N}$, all $\theta_i \in \Theta$, and all $\sigma^n \in \Sigma^n$,

$$\begin{aligned}&\mathbb{E}_{\theta_{-i} \in \Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(S^n\left(\theta_i\right), S^n\left(\theta_{-i}\right)\right), \theta_i\right)\right]\\ \geq &\mathbb{E}_{\theta_{-i} \in \Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(\sigma^n, S^n\left(\theta_{-i}\right)\right), \theta_i\right)\right].\end{aligned}$$

By summing over all $n \in \mathbb{N}$ weighted by agent i's belief about the number of agents, we have

$$\begin{aligned}&\sum_{n \in \mathbb{N}} p_i(n)\, \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(S^n\left(\theta_i\right), S^n\left(\theta_{-i}\right)\right), \theta_i\right)\right]\\ \geq &\sum_{n \in \mathbb{N}} p_i(n)\, \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(\sigma^n, S^n\left(\theta_{-i}\right)\right), \theta_i\right)\right].\end{aligned}$$

Hence, $\left(\Gamma^{\mathbb{N}}, \Sigma^d, S^d\right)$ is dark Bayesian incentive-compatible, where $\Sigma^d = \times_{n \in \mathbb{N}} \Sigma^n$ and $S^d = \left(S^n\right)_{n \in \mathbb{N}}$. By construction, the two mechanisms induce the same outcome rule, and thus they are equivalent.

□

The proof of Proposition 2 highlights the key distinction between lit and dark mechanisms. Lit Bayesian incentive compatibility constraints are satisfied ex-post, meaning they hold for each $n \in \mathbb{N}$ individually. In contrast, dark mechanisms only demands dark Bayesian incentive compatibility constraints—weighted sums of all lit Bayesian incentive compatibility constraints—to be satisfied ex-ante. The converse of Proposition 2 is generally not true. In other words, not every dark mechanism has an equivalent lit mechanism. This discrepancy arises because the strategy space in a dark mechanism may differ from the Cartesian product of the strategy spaces in a lit mechanism, as illustrated in the following example.

Example 3. The dark first-price auction game operates as follows. Each bidder submits a single bid without knowing the number of bidders in the game. The highest bidder wins and pays their bid, with ties broken uniformly at random. In contrast to the (lit) first-price auction game—where bidders of the same type may submit different bids after observing different numbers of bidders—bidders in the dark game must submit the same bid regardless of the number of bidders. Hence, the strategy space Σ^d in the dark first-price auction game is different from the strategy space $\times_{n \in \mathbb{N}} \Sigma^n$ in the (lit) first-price auction game:

$$\Sigma^d = \left\{\sigma^d = \left(\sigma^n\right)_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} \Sigma^n \middle| \forall i, j \in \mathbb{N}, \sigma^i = \sigma^j\right\} \subsetneq \times_{n \in \mathbb{N}} \Sigma^n.$$

Assume that equilibria exist, which we will discuss later. Let β^d denote the equilibrium bidding strategy in the dark first-price auction, and β^n denote the equilibrium bidding strategy in the first-price auction with n bidders. Clearly, β^d fails to satisfy all Bayesian incentive compatibility constraints in the first-price auction with n bidders for all $n \in \mathbb{N}$, unless $\beta^i = \beta^j$ for all $i, j \in \mathbb{N}$, which is generally not the case.

Example 3 shows that the converse of Proposition 2 does not generally hold. It gives a hint of the complexity involved in designing dark mechanisms, arising from the design of the strategy space Σ^d and its interaction with the design of games $\Gamma^{\mathbb{N}}$. This complexity is absent in lit mechanisms, opening new possibilities for the design of dark mechanisms. A key message from Proposition 2 and Example 3 is that the expanded design space for dark mechanisms may enable the implementation of outcome rules that are unattainable in lit mechanisms. In the following, we explore the implications of this message for designing optimal dark auctions. In particular, we examine whether dark auctions generate higher expected revenue compared to lit auctions.

4.2 Optimal Dark Auctions

At first sight, this optimization problem seems quite challenging, as the design space is larger than that of lit auctions. However, an extension of the Revelation Principle (Myerson, 1981) suggests we can restrict our attention, without loss of generality, to a small subset of dark auctions, called "direct dark auctions." We follow the notation in Section 2.2. The proof is relegated to Appendix A.4.

Lemma 2. *Any dark auction $\left(\Gamma^{\mathbb{N}}, \Sigma^d, S^d\right)$ induces an equivalent truth-telling direct dark auction $\left(q^d, t^d, \tilde{S}^d\right)$, where $\tilde{S}^d : \Theta \to \Theta$, $q^d : \Theta^{|N|} \to \mathbb{R}_+^{|N|}$ such that $\sum_{i \in N} q_i^d \leq 1$ and $t^d : \Theta^{|N|} \to \mathbb{R}^{|N|}$ for all $N \in \mathcal{N}$, which implements the same mapping from type profiles to allocations and payments.*

Now we focus on direct dark auctions $\left(q^d, t^d\right)$, where we omit the truth-telling equilibrium strategies.

Let $Q_i^d(\theta_i)$ be the probability of buyer i of type θ_i winning the dark auction, and $Q_i^n(\theta_i)$ be the probability of buyer i of type θ_i winning the dark auction when the number of buyers is $n \in \mathbb{N}$. Then, we have

$$Q_i^d(\theta_i) = \sum_{n=1}^{\infty} p_i(n) \, \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}} \left[q_i^d(\theta_i, \theta_{-i}) \right] = \sum_{n=1}^{\infty} p_i(n) \, Q_i^n(\theta_i),$$

where $p_i(n) = \mathbb{P}\left(|\mathcal{B}| = n \,|\, i \in \mathcal{B}\right)$ is buyer i's belief that the number of buyers is n. Notice that this expression resembles the one for lit auctions, with the key difference being the inclusion of uncertainty regarding the number of buyers.

Let $T_i^d(\theta_i)$ be the expected payment of buyer i of type θ_i in the dark auction, and $T_i^n(\theta_i)$ be the expected payment of buyer i of type θ_i in the dark auction when the number of buyers is n. Then, we have

$$T_i^d(\theta_i) = \sum_{n=1}^{\infty} p_i(n)\, \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[t_i^d(\theta_i, \theta_{-i})\right] = \sum_{n=1}^{\infty} p_i(n)\, T_i^n(\theta_i).$$

Let $U_i^d(\theta_i)$ be the expected utility of buyer i of type θ_i in the dark auction, and $U_i^n(\theta_i)$ be the expected utility of buyer i of type θ_i in the dark auction when the number of buyers is n. Dark Bayesian incentive compatibility implies that, for all $i \in \mathbb{N}$, and all $\theta_i, \theta_i' \in \Theta$,

$$\begin{aligned} U_i^d(\theta_i) = \theta_i Q_i^d(\theta_i) - T_i^d(\theta_i) &= \sum_{n=1}^{\infty} p_i(n)\left(\theta_i Q_i^n(\theta_i) - T_i^n(\theta_i)\right) \\ \geq \theta_i Q_i^d(\theta_i') - T_i^d(\theta_i') &= \sum_{n=1}^{\infty} p_i(n)\left(\theta_i Q_i^n(\theta_i') - T_i^n(\theta_i')\right). \end{aligned}$$

In contrast, lit Bayesian incentive compatibility implies that, for all $i \in \mathbb{N}$, and all $\theta_i, \theta_i' \in \Theta$,

$$U_i^n(\theta_i) = \theta_i Q_i^n(\theta_i) - T_i^n(\theta_i) \geq \theta_i Q_i^n(\theta_i') - T_i^n(\theta_i') \qquad \forall n \in \mathbb{N}.$$

As discussed in Proposition 2 and Example 3, switching from lit auctions to dark auctions leads to a relaxation of incentive compatibility constraints—these constraints hold ex-ante rather than ex-post for each $n \in \mathbb{N}$. As a result, dark auctions implements more outcome rules than lit auctions. The primary question we must address is whether dark auctions yield higher expected revenue than lit auctions.

The answer is no. As first noted by McAfee and McMillan (1987b), dark auctions generate exactly the same highest expected revenue as lit auctions. To ensure clarity and completeness, we provide a proof tailored to our setting (Appendix A.5). We begin by presenting the payoff equivalence lemma for dark auctions.

Lemma 3. *Consider a direct dark auction $\left(q^d, t^d\right)$. Then for all $i \in \mathbb{N}$ and all $\theta_i \in \Theta$, we have*

$$\begin{aligned} U_i^d(\theta_i) &= U_i^d(0) + \int_0^{\theta_i} Q_i^d(x)\, dx, \\ T_i^d(\theta_i) &= T_i^d(0) + \theta_i Q_i^d(\theta_i) - \int_0^{\theta_i} Q_i^d(x)\, dx. \end{aligned}$$

Proof. See Myerson (1981). ☐

Next, we show that the Revenue Equivalence Theorem (Myerson, 1981; Riley and

Samuelson, 1981) extends to dark auctions. The expected payment from buyer i is

$$\begin{aligned}\mathbb{E}_{\theta_i\in\Theta}\left[T_i^d\left(\theta_i\right)\right] &= T_i^d\left(0\right)+\mathbb{E}_{\theta_i\in\Theta}\left[\theta_i Q_i^d\left(\theta_i\right)-\int_0^{\theta_i} Q_i^d\left(x\right)dx\right]\\ &= T_i^d\left(0\right)+\int_0^1\left[\theta_i Q_i^d\left(\theta_i\right)f\left(\theta_i\right)-Q_i^d\left(\theta_i\right)\left(1-F\left(\theta_i\right)\right)\right]d\theta_i\\ &= T_i^d\left(0\right)+\mathbb{E}_{\theta_i\in\Theta}\left[Q_i^d\left(\theta_i\right)v\left(\theta_i\right)\right],\end{aligned}$$

where $v\left(\theta_i\right)=\theta_i-\frac{1-F(\theta_i)}{f(\theta_i)}$. However, to calculate the total expected revenue for the seller, denoted by π^d, we cannot simply sum this expression over all $i\in\mathbb{N}$, because the set of buyers $\mathcal{B}$ is random. Buyer i may belong to the realized set of buyers ($i\in\mathcal{B}$) in some cases, but not in others. When doing the summation, we have to account for that. It turns out that we must weight the expected payment from buyer i exactly by the respective probability that i is included in the realized set of buyers.

Proposition 3 (McAfee and McMillan (1987b))**.** *The expected revenue of a direct dark auction* $\left(q^d,t^d\right)$ *can be expressed as*

$$\begin{aligned}\pi^d &= \sum_{i\in\mathbb{N}}\mathbb{P}\left(i\in\mathcal{B}\right)\mathbb{E}_{\theta_i\in\Theta}\left[T_i^d\left(\theta_i\right)\right]\\ &= \sum_{B\in\mathcal{N}}\mathbb{P}\left(\mathcal{B}=B\right)\mathbb{E}_{\theta_B\in\Theta^{|B|}}\left[\sum_{i\in B}q_i^d\left(\theta_B\right)v\left(\theta_i\right)\right]\\ &\quad+\sum_{i\in\mathbb{N}}\mathbb{P}\left(i\in\mathcal{B}\right)T_i^d\left(0\right).\end{aligned}$$

Ex-post individual rationality requires that $T_i^d(0)\leq 0$. The expected revenue π^d is maximized when $\sum_{i\in B}q_i^d\left(\theta_B\right)v\left(\theta_i\right)$ is maximized for all $B\in\mathcal{N}$, meaning that the dark auction always allocates the item to the buyer of the highest type conditional on being above the reserve price ρ^*. Notice that the dark second-price auction is equivalent to the (lit) second-price auction, because the number of buyers is irrelevant. Optimality (Definition 6) in dark auctions can be achieved by running the dark second-price auction with the reserve price ρ^*, which also attains optimality in lit auctions. Thus, we arrive at the following result.

Corollary 1. *Optimal dark auctions generate the same expected revenue as optimal lit auctions.*

Although dark auctions cannot yield higher expected revenue than lit auctions, the relaxation of incentive compatibility constraints allows for a broader range of outcome rules to achieve optimality—rules that are not implementable in lit auctions. To illustrate this, let us revisit the dark first-price auction in Example 3. To ensure the existence of symmetric equilibrium, we assume that buyers have *common priors* about the number of buyers in the dark auction.[61]

Definition 16. Buyers have *common priors* about the number of buyers in the dark auction if, for all $n \in \mathbb{N}$, there exists $p(n)$ such that $p(n) = \mathbb{P}\left(|\mathcal{B}| = n|\, i \in \mathcal{B}\right)$ for all $i \in \mathbb{N}$.

Now we derive and verify the symmetric equilibrium bidding function $\beta^d : \Theta \to \{0\} \cup [\rho^*, +\infty)$ in the dark first-price auction with the reserve price ρ^*.[62] In a symmetric equilibrium, placing higher bids strictly increases the probability of winning. Dark Bayesian incentive compatibility implies that the probability of winning Q^d is increasing in buyers' types. Therefore, β^d is increasing in buyers' types. Moreover, any symmetric equilibrium bidding function must be strictly increasing on the subdomain of types where $\theta \geq \rho^*$. Otherwise, the auction would end in a tie with probability $\varepsilon > 0$ with several buyers bidding the same amount $b \geq \rho^*$ and each strictly prefers to win at the price b. In this case, a buyer who bids b could increase the expected payoff by bidding slightly more, say $b' > b$. This will increase the probability of winning by at least ε at the cost of $b' - b$, which can be made arbitrarily small, proving that the candidate bidding function is not an equilibrium.[63]

With a strictly increasing symmetric equilibrium bidding function, the dark first-price auction with the reserve price ρ^* induces the same allocation rule as the dark second-price auction with the reserve price ρ^*. Because buyers of type zero obtain payoffs of zero in both auctions, Proposition 3 implies that both auctions are optimal and Lemma 3 ensures that the expected payoffs of all types must be identical in both auctions. Therefore, the only possible symmetric equilibrium bidding function is the one that equates the expected payments in both auctions for all types: for all $\theta_i \geq \rho^*$,

$$\beta^d\left(\theta_i\right) \times \left[\sum_{n=1}^{\infty} p(n) F^{n-1}\left(\theta_i\right)\right] = \sum_{n=1}^{\infty} p(n) \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}}\left[\mathbf{1}_{\theta_i \geq \max\{\theta_{-i}, \rho^*\}} \times \max\left\{\theta_{-i}, \rho^*\right\}\right].$$

The left-hand side is the equilibrium bid times the probability of winning in the dark first-price auction, while the right-hand side is the expected payment in the dark second-price

[61] The exact process of how these common priors are generated is not crucial. To give an example, let $\mathcal{A} = \{1, 2, \ldots, n\}$ be a set of potential buyers. The set of buyers B is generated by selecting each buyer in the set $\mathcal{A}$ independently with the same probability. Then, for all buyers in the set B, they share the same prior about the number of buyers $|B|$.

[62] Without loss of generality, a buyer whose type is below the reserve price ρ^* is assumed to bid zero.

[63] The argument follows the same logic as in Milgrom (2004).

auction. After performing the calculation, we obtain[64]

$$\beta^d(\theta) = \theta - \frac{\int_{\rho^*}^{\theta} \left[\sum_{n=1}^{\infty} p(n) F^{n-1}(x)\right] dx}{\sum_{n=1}^{\infty} p(n) F^{n-1}(\theta)}.$$

We have identified β^d as the unique candidate for the symmetric equilibrium bidding function.[65] It is straightforward to verify that the bidding function β^d constitutes an equilibrium, as it satisfies Lemma 3. (See Myerson (1981) for details.)

Definition 17. Assume common priors. The dark first-price auction game with the reserve price ρ^* admits a unique symmetric equilibrium bidding function β^d. We refer to this game with the strategy β^d as the dark first-price auction with the reserve price ρ^*.

This auction generates the same expected revenue as the (lit) first-price auction with the reserve price ρ^*, as both are optimal. However, they generate different expected revenue for the same realized set of buyers. In other words, they implement different outcome rules to achieve optimality. In particular, the outcome rule implemented by the dark first-price auction cannot be implemented in lit auctions. This is a key advantage of dark auctions. We will see in the next subsection that this advantage can be leveraged to achieve identity compatibility and optimality at the same time.

4.3 Identity Compatibility in Dark Auctions

First, observe that in (direct) dark auctions, the deviation strategy under multiple identities cannot depend on the number of bidders, because the deviator no longer observes it.[66] Accordingly, we impose that $\hat{\theta}_{N_i}^{|B|} = \hat{\theta}_{N_i}$ and $\hat{\theta}_S^{|B|} = \hat{\theta}_S$ for all $|B| \in \mathbb{N}$ in Bayesian identity compatibility for dark auctions (Definition 9 and 11). We omit the formal definitions for brevity. Apart from this adjustment, all notions of identity compatibility for lit auctions carry over directly to dark auctions.[67]

[64] We define $F^0(\cdot) = 1$.

[65] Notice that the equilibrium bidding function reduces to the conventional one when the distribution of the number of buyers degenerates.

[66] This is because we focus on direct auctions. In indirect dark auctions, the deviation strategy under multiple identities could possibly depend on the number of bidders. However, restricting attention to direct auctions—whether lit or dark—entails no loss of generality. See Appendix 5.3 for more details.

[67] Ex-post identity compatibility remains unchanged in dark auctions, as the type profile itself reveals the number of bidders (Definition 10, 13, and 14).

4.3.1 Buyer Identity Compatibility

We start with buyer identity compatibility. The results from lit auctions extend to dark auctions. It is straightforward to verify that the dark first-price auction is Bayesian buyer identity-compatible, as discussed in Example 1. Take the optimal one for example. We have the allocation and payment rules as follows:

$$
\begin{aligned}
q_i^{d-1st}(\theta_i, \theta_{-i}) &= \mathbf{1}_{\theta_i > \max\{\theta_{-i}, \rho^*\}}, \\
t_i^{d-1st}(\theta_i, \theta_{-i}) &= \mathbf{1}_{\theta_i > \max\{\theta_{-i}, \rho^*\}} \times \beta^d(\theta_i).
\end{aligned}
$$

Bayesian buyer identity compatibility for dark auctions requires us to check that

$$
\begin{aligned}
&\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\mathbf{1}_{\theta_i > \max\{\theta_{-i}, \rho^*\}} \times \left(\theta_i - \beta^d(\theta_i)\right)\right] \middle| i \in B\right] \\
\geq &\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\mathbf{1}_{\max\left\{\hat{\theta}_{N_i}\right\} > \max\{\theta_{-i}, \rho^*\}} \times \left(\theta_i - \beta^d\left(\max\left\{\hat{\theta}_{N_i}\right\}\right)\right)\right] \middle| i \in B\right],
\end{aligned}
$$

which is guaranteed by dark Bayesian incentive compatibility of the dark first-price auction (with reserve). The equality holds when $\max\left\{\hat{\theta}_{N_i}\right\} = \theta_i$, and the number of identities $|N_i|$ is irrelevant.

Ex-post buyer identity compatibility still implies strategy-proofness in dark auctions (Lemma 1). Since strategy-proofness makes the number of bidders irrelevant, the proof of Theorem 1 remains valid even when the number of bidders is concealed. Thus, the conclusion still holds in dark auctions that the dark second-price auction with (or without) the reserve price ρ^* is the unique optimal (or optimally efficient) dark auction that is ex-post buyer identity-compatible.

4.3.2 Seller Identity Compatibility

Notice that the optimal reserve price remains unchanged regardless of the number of buyers and is therefore unaffected by whether the number of bidders is disclosed. Consequently, Proposition 1 remains true that the dark second-price auction with the reserve price ρ^* is Bayesian seller identity-compatible.

Returning to our motivation for introducing dark auctions, we seek optimal auctions that are ex-post seller identity-compatible, given the impossibility result of Theorem 2 for lit auctions. We now show that the dark first-price auction with the reserve price ρ^* satisfies this criterion. In fact, we can establish a stronger result: it is ex-post auctioneer identity-compatible. It suffices to verify that

$$\begin{aligned}&\mathbb{E}_{\theta_B\in\Theta^{|B|}}\left[\sum_{i\in B}\mathbf{1}_{\theta_i>\max\{\theta_{-i},\rho^*\}}\times\beta^d\left(\theta_i\right)\right]\\ \geq&\mathbb{E}_{\theta_B\in\Theta^{|B|}}\left[\sup_{\theta_S\in\Theta^{|S|}}\sum_{i\in B}\mathbf{1}_{\theta_i>\max\{\theta_{-i},\theta_S,\rho^*\}}\times\beta^d\left(\theta_i\right)\right],\end{aligned}$$

which holds because both sides simplify to $\mathbb{E}_{\theta_B\in\Theta^{|B|}}\left[\mathbf{1}_{\max\{\theta_B\}>\rho^*}\times\beta^d\left(\max\left\{\theta_B\right\}\right)\right]$.

Note that it is not unique. For instance, we can construct a fixed payment scheme, $\beta^d\left(\theta,n\right)$, for each realized set of buyers B with $|B|=n$. Let $\beta^d\left(\theta,n\right)$ decrease with the number of buyers, n, while maintaining dark Bayesian incentive compatibility by ensuring that its expected sum equals $\beta^d\left(\theta\right)$, i.e., $\sum_{n=1}^{\infty}p\left(n\right)\beta^d\left(\theta,n\right)=\beta^d\left(\theta\right)$. The seller is always worse off by participating in such an auction, making it ex-post auctioneer identity-compatible while remaining optimal. Although such alternative mechanisms exist, we argue that the dark first-price auction is the simplest and most commonly observed in practice. Furthermore, if these alternative auctions were implemented, buyers would find it profitable to manipulate payments by inflating their identities. This leads to the following characterization of the dark first-price auction. The proof is relegated to Appendix A.6.

Theorem 4. *Assume common priors. The dark first-price auction with the reserve price ρ^* is the unique optimal dark auction that is ex-post auctioneer identity-compatible and Bayesian buyer identity-compatible. The dark first-price auction is the unique optimally efficient dark auction that is ex-post auctioneer identity-compatible and Bayesian buyer identity-compatible.*

To understand how dark auctions circumvent the conflict between revenue maximization and the creation of fake competition through identities in lit auctions, as highlighted in Theorem 2, we must first examine the impact of shill bidding on competition in the auction. Shill bidding offers participants two key avenues for manipulation: first, by altering the auction format through the selection of a specific number of bidders (identities), and second, by optimizing the strategies of the bidders they control within the specific auction. Similarly, competition in the auction can be analyzed from two perspectives: extensive competition, which arises from variations in the number of bidders, and intensive competition, which pertains to a fixed number of bidders. Different auctions exhibit different combinations of these two forms of competition. In the second-price auction, extensive competition is subsumed by intensive competition, which can be exploited by the seller. In contrast, the first-price auction integrates the intensive competition into a fixed bid, making it resistant to manipulation by other participants. However, the extensive competition remained in the first price auction can still be exploited by the seller, as the equilibrium bidding function increases with the number of bidders. The dark first-price auction addresses this by consolidating the extensive competition into a fixed bid across different numbers of bidders, thereby muting the channel of intensifying the perceived competition among buyers by leveraging more identities (bidders) in the auction and preventing such exploitation.[68]

As a real-world example, Google transitioned from the second-price auction to the first-price auction for online advertising auctions in 2019.[69] Notably, what Google refers to as the first-price auction is actually the dark first-price auction, as the number of advertisers is typically not disclosed. Google is currently facing a lawsuit over alleged shill-bidding-like practices in these auctions (Footnote 6). Some of the allegations relate specifically to the second-price auction and no longer apply to the dark first-price auction. This shift has at least eased concerns among market participants about shill bidding.

Information disclosure policy is a crucial aspect of auction design in practice.[70] In particular, it can facilitate collusion by providing a mechanism for signalling and punishment (Cramton and Schwartz, 2000; Klemperer, 2002, 2003).[71] The prevailing view is that too much transparency might hurt. For example, OECD guidelines for fighting bid rigging in public procurement state that "Transparency requirements are indispensable for a sound procurement procedure to aid in the fight against corruption. They should be complied with in a balanced manner, in order not to facilitate collusion by disseminating information beyond legal requirements.... Limit as much as possible communications between bidders during the tender process. Open tenders enable communication and signalling between bidders" (OECD, 2009).

This paper focuses specifically on the disclosure of the number of bidders. The results align with the conventional wisdom—but for a clear and straightforward reason: if the disclosed information can be falsified by fake identities to mislead participants, it is better to conceal it.

[68] In a different context, without concerns about shill bidding but maintaining uncertainty about the number of buyers, McAfee and McMillan (1987b) and Matthews (1987) demonstrate that the seller can actually benefit from concealing the number of buyers when buyers exhibit constant or decreasing absolute risk aversion. In particular, they compare the first-price auction, the second-price auction, and the first-price auction concealing the number of buyers. Assuming constant or decreasing absolute risk aversion, the expected revenue is highest in the first-price auction concealing the number of buyers. For experimental evidence, see Dyer, Kagel and Levin (1989) and Aycinena and Rentschler (2018). In other words, the dark first-price auction proves to be appealing both to buyers and, to some degree, to the seller.

[69] An Update on First Price Auctions for Google Ad Manager, May 10, 2019.

[70] By the Revenue Equivalence Theorem (Myerson, 1981), information disclosure is theoretically irrelevant as long as the same outcome rule is implemented.

[71] See Footnote 15.

5 Discussion

5.1 Identity Compatibility for Both Parties

So far, we have primarily analyzed identity compatibility for buyers and the seller separately. It should come as no surprise that achieving identity compatibility for both parties simultaneously is challenging, as their incentives to engage in shill bidding are misaligned. As a result, we have the following auction dilemma:

Proposition 4. *No optimal dark auction can be both ex-post buyer and ex-post seller identity-compatible. No optimally efficient dark auction can be both ex-post buyer and ex-post seller identity-compatible.*

Proof. Ex-post buyer identity compatibility implies strategy-proofness (Lemma 1). Optimality (or optimal efficiency) and strategy-proofness pin down the second-price auction with (or without) reserve. However, the second-price auction is not ex-post seller identity-compatible. Therefore, no optimal (or optimally efficient) auction can be both ex-post buyer and ex-post seller identity-compatible. □

This impossibility result also applies to lit auctions, as dark auctions encompass lit auctions (Proposition 2). The intuition is that ex-post buyer identity compatibility requires the winning payment to be independent of the winner's type. Meanwhile, ex-post seller identity compatibility requires the winning payment to be independent of the loser's type; otherwise, the seller could manipulate the payment by pretending to be a loser in the auction. As a result, the winning payment must be fixed, which contradicts optimality (or optimal efficiency).

Despite that ex-post identity compatibility for both buyers and the seller is unattainable, we can still achieve a weaker form of identity compatibility for both parties simultaneously. Theorem 1 and Proposition 1 show that ex-post buyer identity compatibility, Bayesian seller identity compatibility, and optimality can be attained simultaneously by the second-price auction with the reserve price ρ^*. Furthermore, Theorem 4 demonstrates that we can achieve Bayesian buyer identity compatibility, ex-post seller identity compatibility, and efficiency (or optimality) simultaneously by running the dark first-price auction (or with reserve). This underscores a tradeoff between buyers and the seller, where the choice of auction depends on which party is more informed when shill bidding.

5.2 Disclosure Policy

This paper demonstrates the role of concealing the number of bidders in the presence of shill bidding. Notably, disclosure is not necessarily an all-or-nothing decision; instead, the designer has the option to partially reveal this information. For example, the designer can run the second-price auction when there are no more than five bidders, and switch to the dark first-price auction otherwise. This mechanism effectively signals whether the number of bidders is below or above five. More generally, we define a *disclosure policy* ϕ as a surjective mapping from the number of agents $\mathbb{N}$ to a set of signals Y.[72] The above example illustrates a *partitional* disclosure policy.

Definition 18. A disclosure policy ϕ is *partitional* if, for any $y, y' \in Y$, $\phi^{-1}(y) \cap \phi^{-1}(y') \neq \emptyset$ implies $y = y'$.

Lit and dark mechanisms are both examples of mechanisms under partitional disclosure policies, where $Y^{\text{lit}} = \mathbb{N}, \phi^{\text{lit}} = \text{id}_{\mathbb{N}}$ and $Y^{\text{dark}} = \{d\}, \phi^{\text{dark}} \equiv d$.

We follow the notation in Section 4.1. Consider a collection of games $\Gamma^{\mathbb{N}} = (\Gamma^n)_{n \in \mathbb{N}}$ and a partitional disclosure policy ϕ. When agents observe a signal $y \in Y$, the set of strategies for each agent under signal y is denoted by $\Sigma^y \subseteq \times_{n \in \phi^{-1}(y)} \Sigma^n$, with generic element $\sigma^y = (\sigma^n)_{n \in \phi^{-1}(y)}$. As noted earlier, Σ^y does not have to be identical to $\times_{n \in \phi^{-1}(y)} \Sigma^n$. Under each signal y, a type-strategy $S^y = (S^n)_{n \in \phi^{-1}(y)}$ for each agent maps from types to strategies, i.e., $S^y : \Theta \to \Sigma^y$. We characterize mechanisms under partitional disclosure policies as follows.

Definition 19. $\left\{\Gamma^{\mathbb{N}}, (\Sigma^y, S^y)_{y \in Y}, \phi\right\}$ is *partitional Bayesian incentive-compatible* if, for all $i \in \mathbb{N}$, all $\theta_i \in \Theta$, and all $y \in Y$,

$$S^y(\theta_i) \in \arg\max_{\sigma^y \in \Sigma^y} \sum_{n \in \phi^{-1}(y)} p_i(n|\, y)\, \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}} \left[u_i \left(g^{\Gamma^n} (\sigma^n, S^n(\theta_{-i})), \theta_i \right) \right],$$

where $p_i(n|\, y) = \mathbb{P}(|\mathcal{B}| = n|\, i \in \mathcal{B}, y)$ is agent i's belief that the number of agents is n under signal y.

We refer to $\left\{\Gamma^{\mathbb{N}}, (\Sigma^y, S^y)_{y \in Y}, \phi\right\}$ as a mechanism under the partitional disclosure policy ϕ—or, more concisely, a *partitional mechanism*—if it is partitional Bayesian incentive-compatible. The term "partitional" emphasizes that the information about the number of agents participating in the game is revealed according to a partitional disclosure policy. It is straightforward to verify that this definition reduces to lit Bayesian incentive compatibility (Definition 2) under the disclosure policy ϕ^{lit}, and to dark Bayesian incentive

[72]It is without loss to focus on surjective mappings, because we can ignore unused signals.

compatibility (Definition 15) under the disclosure policy ϕ^{dark}. Moreover, our earlier observation (Proposition 2) extends to partitional disclosure policies. The proof is relegated to Appendix A.7.

Theorem 5. *Any partitional mechanism induces an equivalent dark mechanism.*

This implies that any outcome rule implementable in a mechanism under a partitional disclosure policy can be implemented in a dark mechanism. In other words, dark mechanisms are the most general class of mechanisms under partitional disclosure policies. Therefore, when we analyze identity compatibility in dark auctions, we implicitly take into account all auctions under partitional disclosure policies. Theorem 4 can be restated as characterizing the dark first-price auction (with reserve) as the unique optimal auction under any partitional disclosure policy that is ex-post auctioneer identity-compatible and Bayesian buyer identity-compatible.

Under a non-partitional disclosure policies $\hat{\phi}$, there exist two distinct signals, y and y' such that $\hat{\phi}^{-1}(y) \cap \hat{\phi}^{-1}(y') \neq \emptyset$. Suppose $k \in \hat{\phi}^{-1}(y) \cap \hat{\phi}^{-1}(y')$. This implies that agents must decide which strategies to play in the game Γ^k under both signals y and y'. In particular, agents may choose different strategies for the same game Γ^k depending on the signal, because the set of games induced by signal y, y' are different in general, i.e., $(\Gamma^n)_{n \in \hat{\phi}^{-1}(y)} \neq (\Gamma^n)_{n \in \hat{\phi}^{-1}(y')}$. The strategy space under signal y, $\Sigma^y \subseteq \times_{n \in \hat{\phi}^{-1}(y)} \Sigma^n$, also differs from that under signal y', $\Sigma^{y'} \subseteq \times_{n \in \hat{\phi}^{-1}(y')} \Sigma^n$. Put differently, non-partitional disclosure policies introduce correlation among agents' strategies for the same game Γ^k that is absent in partitional disclosure policies. This correlation is beyond the scope of our analysis, but it is an interesting avenue for future research.

5.3 Implementation via Extensive Forms

While the paper focuses on outcome rules (or direct mechanisms), it is important to recognize that any outcome rule can, in principle, be implemented through dynamic mechanisms via extensive forms (or indirect mechanisms). In dynamic mechanisms, on-path strategies can be fully characterized by type profiles; however, off-path strategies may also emerge, which cannot be captured solely through type profiles. Hence, in any dynamic mechanism, agents can replicate the behavior of the type they would have reported in the direct mechanism. Furthermore, dynamic mechanisms enable agents to explore off-path deviations, which may become profitable when leveraging multiple identities. The presence of off-path deviations in dynamic auctions strengthens the concept of identity compatibility, as the expanded strategy space available in dynamic auctions offers participants more opportunities to manipulate outcomes through the use of multiple identities compared to direct auctions. As a result, if a dynamic auction satisfies a specific notion of identity compatibility as defined in this paper, the corresponding direct auction must also satisfy the same notion.

The converse, however, does not hold in general. As illustrated in Example 2, the fact that off-path deviations are suboptimal when a buyer uses a single identity does not imply their suboptimality when a buyer can simultaneously employ multiple identities (see Footnote 46). Nonetheless, when the objective is to design auctions that disincentive shill bidding, it is without loss of generality to restrict attention to direct auctions (or outcome rules). That said, care must be taken when considering alternative implementations of a given outcome rule, as indirect implementations may introduce deviations that are not possible under direct implementations. Since all results in this paper are formulated in terms of outcome rules, they remain valid when considering dynamic mechanisms.[73]

The literature often adopts the finite type space for technical convenience when analyzing dynamic auctions through finite extensive forms (Akbarpour and Li, 2020; Komo, Kominers and Roughgarden, 2024).[74] This paper does not explicitly model the extensive-form game and therefore does not focus on the finite extensive-form game with the finite type space. Nevertheless, as a robustness check, we extend our analysis to the finite type space. The key findings are summarized below, with complete proofs provided in Online Appendix B.1.

Our main insights continue to hold in the finite type space. In particular, lit auctions allow the seller to heighten the perceived competition by inflating the number of bidders, whereas dark auctions mute the channel. This channel operates independently of the choice of type space. Consequently, the impossibility result (Theorem 2) for lit auctions remains valid because of the conflict between revenue maximization and fake competition. From a technical perspective, we need to handle ties carefully in the finite type space, since participants can submit identical bids under multiple identities to increase their chances of winning ties or affect the payments. As is well-documented in the literature, the domain of the type space can affect characterization results. For instance, strategy-proofness and (optimal) efficiency do not uniquely pin down the second-price auction (Holmström, 1979; Harris and Raviv, 1981; Lovejoy, 2006; Elkind, 2007; Jeong and Pycia, 2023). In our setting, the characterization of the dark first-price auction (Theorem 4) does not extend to the finite type space.[75] Interestingly, the second-price auction can still be characterized in the

[73]The observation that deviation is generally easier in dynamic (or indirect) mechanisms than in direct mechanisms is also noted by Levin and Peck (2023), who refer to this phenomenon as the Misbehavior Principle.

[74]See Simon and Stinchcombe (1989) for a discussion of modeling challenges involved in the continuous-time game theory.

[75]In the finite type space, the dark first-price auction with (or without) reserve remains optimal (or optimally efficient) and ex-post auctioneer identity-compatible, because the channel through which the seller heightens the perceived competition is shut down. The characterization fails only because

finite type space as follows, underscoring the distinction between ex-post buyer identity compatibility and strategy-proofness.

Theorem 6. *The second-price auction maximizes expected revenue among all efficient auctions that are ex-post buyer identity-compatible.*

Finally, the characterization of the posted-price mechanism (Theorem 3) remains valid in the finite type space, as ties have already been considered in this case. All discussions about general mechanisms remain valid regardless of the type space—for example, Theorem 5.

5.4 Distinguishable Agents

The main purpose of this paper is to study shill bidding in auctions. The basic framework assumes that agents are indistinguishable, because we want to model the scenario in which the designer is unable to distinguish between genuine buyers and shill bidders. However, it is worth noting that the framework can be naturally extended to environments with distinguishable agents, where agents are not necessarily ex-ante identical. In such cases, the collection of games to be designed is no longer inherently anonymous and may depend on the specific identities of the agents.

This extension introduces a slightly different design consideration: the disclosure of agents' identities, as opposed to merely the number of agents. Under this perspective, lit mechanisms fully disclose the identities of agents, whereas dark mechanisms conceal this information. In Online Appendix B.2, we formally define this extended framework and establish that Theorem 5 continues to hold when agents are distinguishable. In particular, dark mechanisms remain the most general class of mechanisms under partitional disclosure policies, in the sense that they implement the widest range of outcome rules. This paper demonstrates how this principle can help the designer to circumvent the inherent conflict between revenue maximization and fake competition in lit auctions. We leave it for future explorations that how to apply this principle to other environments.

References

Akbarpour, Mohammad and Shengwu Li (2020) "Credible Auctions: A Trilemma," *Econometrica*, 88 (2), 425–467, 10.3982/ECTA15925. 2, 8, 21, 24, 38, 59, 60

Andreyanov, Pavel, Jernej Čopič, and Byeong-Hyeong Jeong (2018) "Optimal Robust

buyers can increase their chances of winning ties by employing multiple identities.

Divisible Mechanisms for Public Goods," *Working Paper*, https://pandreyanov.github.io/files/OptimalRobustPublicGoods.pdf. 8

Arnosti, Nick, Marissa Beck, and Paul Milgrom (2016) "Adverse Selection and Auction Design for Internet Display Advertising," *American Economic Review*, 106 (10), 2852–2866, 10.1257/aer.20141198. 7

Ausubel, Lawrence M. and Paul Milgrom (2002) "Ascending Auctions with Package Bidding," *The B.E. Journal of Theoretical Economics*, 1 (1), 10.2202/1534-5963.1019. 7

——— (2005) "The Lovely but Lonely Vickrey Auction," in Cramton, Peter, Yoav Shoham, and Richard Steinberg eds. *Combinatorial Auctions*, 17–40: The MIT Press, 10.7551/mitpress/9780262033428.003.0002. 7

Aycinena, Diego and Lucas Rentschler (2018) "Auctions with Endogenous Participation and an Uncertain Number of Bidders: Experimental Evidence," *Experimental Economics*, 21 (4), 924–949, 10.1007/s10683-017-9558-8. 34

Bulow, Jeremy and Paul Klemperer (1996) "Auctions Versus Negotiations," *The American Economic Review*, 86 (1), 180–194, https://www.jstor.org/stable/2118262. 3

Cassady, Ralph (1967) *Auctions and Auctioneering*: University of California Press, 1st edition, 10.2307/jj.2430697. 2

Chakraborty, Indranil and Georgia Kosmopoulou (2004) "Auctions with Shill Bidding," *Economic Theory*, 24 (2), 271–287, 10.1007/s00199-003-0423-y. 7

Chassang, Sylvain and Juan Ortner (2019) "Collusion in Auctions with Constrained Bids: Theory and Evidence from Public Procurement," *Journal of Political Economy*, 127 (5), 2269–2300, 10.1086/701812. 17

Chen, Kong-Pin, T. P. Liang, Ted Chang, Yi-chun Liu, Shou-Yung Yin, and Ya-Ting Yu (2020) "How Serious Is Shill Bidding in Online Auctions? Evidence from eBay Motors," September, 10.2139/ssrn.3279292. 2

Clarke, Edward H. (1971) "Multipart Pricing of Public Goods," *Public Choice*, 11, 17–33, https://www.jstor.org/stable/30022651. 7

Čopič, Jernej and Clara Ponsatí (2016) "Optimal Robust Bilateral Trade: Risk Neutrality," *Journal of Economic Theory*, 163, 276–287, 10.1016/j.jet.2015.11.008. 8

Cramton, Peter and Jesse A. Schwartz (2000) "Collusive Bidding: Lessons from the FCC Spectrum Auctions," *Journal of Regulatory Economics*, 17 (3), 229–252, 10.1023/A:1008174031940. 6, 34

Dworczak, Piotr (2020) "Mechanism Design With Aftermarkets: Cutoff Mechanisms," *Econometrica*, 88 (6), 2629–2661, 10.3982/ECTA15768. 23

Dyer, Douglas, John H. Kagel, and Dan Levin (1989) "Resolving Uncertainty about the Number of Bidders in Independent Private- Value Auctions: An Experimental Analysis," *The RAND Journal of Economics*, 20 (2), 268–279, 10.2307/2555693. 34

Elkind, Edith (2007) "Designing and Learning Optimal Finite Support Auctions," in *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '07, 736–745, New Orleans, Louisiana, January, https://dl.acm.org/doi/10.5555/1283383.1283462. 5, 38, 58, 65, 68, 69, 70, 74

Graham, Daniel A. and Robert C. Marshall (1987) "Collusive Bidder Behavior at Single-Object Second-Price and English Auctions," *Journal of Political Economy*, 95 (6), 1217–1239, https://www.jstor.org/stable/1831119. 17

Graham, Daniel A., Robert C. Marshall, and Jean-Francois Richard (1990) "Phantom Bidding against Heterogeneous Bidders," *Economics Letters*, 32 (1), 13–17, 10.1016/0165-1765(90)90043-Z. 7

Green, Jerry and Jean-Jacques Laffont (1977) "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," *Econometrica*, 45 (2), 427–438, 10.2307/1911219. 8, 19, 58

Groves, Theodore (1973) "Incentives in Teams," *Econometrica*, 41 (4), 617–631, 10.2307/1914085. 7

Häfner, Samuel, Marek Pycia, and Haoyuan Zeng (2024) "Mechanism Design with Information Leakage," *Working Paper*. 8

Hagerty, Kathleen M and William P Rogerson (1987) "Robust Trading Mechanisms," *Journal of Economic Theory*, 42 (1), 94–107, 10.1016/0022-0531(87)90104-9. 8

Harris, Milton and Artur Raviv (1981) "Allocation Mechanisms and the Design of Auctions," *Econometrica*, 49 (6), 1477–1499, 10.2307/1911413. 5, 38, 58

Harstad, Ronald M., John H. Kagel, and Dan Levin (1990) "Equilibrium Bid Functions for Auctions with an Uncertain Number of Bidders," *Economics Letters*, 33 (1), 35–40, 10.1016/0165-1765(90)90197-9. 8

Herrera, Carolina (2023) "NYC Auction Houses — One Year After Deregulation," *The Columbia Journal of Law & the Arts*, `https://journals.library.columbia.edu/index.php/lawandarts/announcement/view/667`. 2

Holmström, Bengt (1979) "Groves' Scheme on Restricted Domains," *Econometrica*, 47 (5), 1137–1144, 10.2307/1911954. 5, 8, 19, 38, 58

Holt, Charles A. (1979) "Uncertainty and the Bidding for Incentive Contracts," *The American Economic Review*, 69 (4), 697–705, `https://www.jstor.org/stable/1808719`. 3

Hurwicz, Leonid (1972) "On Informationally Decentralized Systems," in C. B. McGuire and Roy Radner eds. *Decision and Organization: A Volume in Honor of Jacob Marschak*, 297–336, Amsterdam and London: North-Holland Pub. Co. 5

Izmalkov, Sergei (2004) "Shill Bidding and Optimal Auctions," in *The 15th Stony Brook International Conference on Game Theory*, Stony Brook, New York, July. 7, 21

Jeong, Byeong-hyeon and Marek Pycia (2023) "The First-Price Principle of Maximizing Economic Objectives," *Working Paper*, `https://sites.econ.uzh.ch/pycia/jeong-pycia-principle.pdf`. 5, 8, 38, 58

Klemperer, Paul (1998) "Auctions with Almost Common Values: The 'Wallet Game' and Its Applications," *European Economic Review*, 42 (3), 757–769, 10.1016/S0014-2921(97)00123-2. 17

——— (2002) "What Really Matters in Auction Design," *The Journal of Economic Perspectives*, 16 (1), 169–189, `https://www.jstor.org/stable/2696581`. 3, 6, 34

——— (2003) "Using and Abusing Economic Theory," *Journal of the European Economic Association*, 1 (2-3), 272–300, 10.1162/154247603322390937. 6, 34

Komo, Andrew, Scott Duke Kominers, and Tim Roughgarden (2024) "Shill-Proof Auctions," November, 10.48550/arXiv.2404.00475. 8, 23, 38, 61

Lamy, Laurent (2009) "The Shill Bidding Effect versus the Linkage Principle," *Journal of Economic Theory*, 144 (1), 390–413, 10.1016/j.jet.2008.06.001. 7

——— (2013) "“Upping the Ante”: How to Design Efficient Auctions with Entry?" *The RAND Journal of Economics*, 44 (2), 194–214, 10.1111/1756-2171.12017. 7

Lauermann, Stephan and Andre Speit (2023) "Bidding in Common-Value Auctions With an Unknown Number of Competitors," *Econometrica*, 91 (2), 493–527, 10.3982/ECTA17793. 9

Levin, Dan and James Peck (2023) "Misbehavior in Common Value Auctions: Bidding Rings and Shills," *American Economic Journal: Microeconomics*, 15 (1), 171–200, 10.1257/mic.20200447. 7, 38

Li, Shengwu (2017) "Obviously Strategy-Proof Mechanisms," *American Economic Review*, 107 (11), 3257–3287, 10.1257/aer.20160425. 8

Lovejoy, William S. (2006) "Optimal Mechanisms with Finite Agent Types," *Management Science*, 52 (5), 788–803, `https://www.jstor.org/stable/20110553`. 5, 38, 58, 65, 68, 69, 70, 74

Mailath, George J and Peter Zemsky (1991) "Collusion in Second Price Auctions with Heterogeneous Bidders," *Games and Economic Behavior*, 3 (4), 467–486, 10.1016/0899-8256(91)90016-8. 17

Matthews, Steven (1987) "Comparing Auctions for Risk Averse Buyers: A Buyer's Point of View," *Econometrica*, 55 (3), 633–646, 10.2307/1913603. 8, 34

McAfee, R. Preston and John McMillan (1987a) "Auctions and Bidding," *Journal of Economic Literature*, 25 (2), 699–738, `https://www.jstor.org/stable/2726107`. 3

——— (1987b) "Auctions with a Stochastic Number of Bidders," *Journal of Economic Theory*, 43 (1), 1–19, 10.1016/0022-0531(87)90113-X. 8, 28, 29, 34

——— (1992) "Bidding Rings," *The American Economic Review*, 82 (3), 579–599, `https://www.jstor.org/stable/2117323`. 17

Milgrom, Paul (2004) *Putting Auction Theory to Work*, Churchill Lectures in Economics, Cambridge: Cambridge University Press, 10.1017/CBO9780511813825. 3, 30

——— (2017) *Discovering Prices: Auction Design in Markets with Complex Constraints*, New York: Columbia University Press. 3

Milgrom, Paul R. and Robert J. Weber (1982) "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50 (5), 1089–1122, 10.2307/1911865. 3, 7, 14

Myerson, Roger B. (1981) "Optimal Auction Design," *Mathematics of Operations Research*, 6 (1), 58–73, https://www.jstor.org/stable/3689266. 3, 4, 14, 15, 27, 28, 31, 34, 45, 47, 49, 58, 70

OECD (2009) *Guidelines for Fighting Bid Rigging in Public Procurement*, Paris: OECD Publishing, 10.1787/8cfeafbb-en. 34

Pycia, Marek and Madhav Raghavan (2021) "Non-Bossiness and First-Price Auctions," June, 10.2139/ssrn.3941784. 8

Pycia, Marek and Peter Troyan (2023) "A Theory of Simplicity in Games and Mechanism Design," *Econometrica*, 91 (4), 1495–1526, 10.3982/ECTA16310. 8

Riley, John G. and William F. Samuelson (1981) "Optimal Auctions," *The American Economic Review*, 71 (3), 381–392, https://www.jstor.org/stable/1802786. 28

Robinson, Marc S. (1985) "Collusion and the Choice of Auction," *The RAND Journal of Economics*, 16 (1), 141–145, 10.2307/2555595. 17

Roth, Alvin E. and Axel Ockenfels (2002) "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review*, 92 (4), 1093–1103, 10.1257/00028280260344632. 14

Sakurai, Yuko, Makoto Yokoo, and Shigeo Matsubara (1999) "A Limitation of the Generalized Vickrey Auction in Electronic Commerce: Robustness against False-Name Bids," in *Proceedings of the AAAI Conference on Artificial Intelligence*, 86–92, Orlando, Florida, July, https://aaai.org/proceeding/aaai-16-1999/. 7

Seibel, Regina and Samuel Škoda (2023) "Collusion by Exclusion in Public Procurement," *Working Paper*, https://reginaseibel.github.io/publication/collusion/collusion.pdf. 2, 17, 18

Sher, Itai (2012) "Optimal Shill Bidding in the VCG Mechanism," *Economic Theory*, 50 (2), 341–387, 10.1007/s00199-010-0566-6. 7

Shinozaki, Hiroki (2024) "Shill-Proof Rules in Object Allocation Problems with Money," *Working Paper*, https://ideas.repec.org/p/hit/hiasdp/hias-e-137.html. 7, 8, 24

Simon, Leo K. and Maxwell B. Stinchcombe (1989) "Extensive Form Games in Continuous Time: Pure Strategies," *Econometrica*, 57 (5), 1171–1214, 10.2307/1913627. 38

Vickrey, William (1961) "Counterspeculation, Auctions, and Competitive Sealed Tenders," *The Journal of Finance*, 16 (1), 8–37, 10.2307/2977633. 3, 7, 14

Wang, Wenli, Hidvégi, Zoltán, and Andrew B Whinston (2004) "Shill-Proof Fee (SPF) Schedule: The Sunscreen against Seller Self-Collusion in Online English Auctions," *Working Paper*. 8

Yokoo, Makoto, Yuko Sakurai, and Shigeo Matsubara (2004) "The Effect of False-Name Bids in Combinatorial Auctions: New Fraud in Internet Auctions," *Games and Economic Behavior*, 46 (1), 174–188, 10.1016/S0899-8256(03)00045-9. 7

A Proofs Omitted from the Main Text

A.1 Proof of Theorem 1

Note that the second-price auction is efficient and becomes optimal when the reserve price ρ^* is applied (Myerson, 1981). We first verify that the second-price auction with any reserve price ρ is ex-post buyer identity-compatible. In particular, we show that no buyer $i \in B$ can benefit from using multiple identities for all $B \in \mathcal{N}$, even if buyer i does not commit to the number of identities in advance. There are three cases:

1. $\theta_i > \max\{\theta_{-i}, \rho\}$: Buyer i wins the auction with the payment $\max\{\theta_{-i}, \rho\}$ when bidding with only one identity, and obtains a positive payoff $\theta_i - \max\{\theta_{-i}, \rho\} > 0$. We now show that i cannot improve by using multiple identities. Following the notation in Definition 10, for any $\hat{\theta}_{N_i} \in \Theta^{|N_i|}$, if $\max\left\{\hat{\theta}_{N_i}\right\} \geq \max\{\theta_{-i}, \rho\}$, buyer i can at best[76] win the auction with the same payment as before; if $\max\left\{\hat{\theta}_{N_i}\right\} < \max\{\theta_{-i}, \rho\}$, buyer i loses the auction with a zero payoff. Hence, buyer i cannot benefit from using multiple identities in any case when $\theta_i > \max\{\theta_{-i}, \rho\}$.

2. $\theta_i = \max\{\theta_{-i}, \rho\}$: Conditional on winning, buyer i pays θ_i when bidding with only one identity, and obtains a zero payoff. For any $\hat{\theta}_{N_i} \in \Theta^{|N_i|}$, if $\max\left\{\hat{\theta}_{N_i}\right\} \geq \max\{\theta_{-i}, \rho\}$, buyer i can only win the auction with the same payment and obtains a zero payoff; if $\max\left\{\hat{\theta}_{N_i}\right\} < \max\{\theta_{-i}, \rho\}$, buyer i loses the auction with a zero payoff. Hence, buyer i cannot benefit from using multiple identities in any case when $\theta_i = \max\{\theta_{-i}, \rho\}$.

3. $\theta_i < \max\{\theta_{-i}, \rho\}$: Buyer i loses the auction when bidding with only one identity, and obtains a zero payoff. For any $\hat{\theta}_{N_i} \in \Theta^{|N_i|}$, if $\max\left\{\hat{\theta}_{N_i}\right\} \geq \max\{\theta_{-i}, \rho\}$, buyer i can

[76] When $\max\left\{\hat{\theta}_{N_i}\right\} = \max\{\theta_{-i}\}$, buyer i will not win the auction with probability one.

only win the auction with the payment $\max\{\theta_{-i},\rho\}$ and obtains a negative payoff $\theta_i - \max\{\theta_{-i},\rho\} < 0$; if $\max\left\{\hat{\theta}_{N_i}\right\} < \max\{\theta_{-i},\rho\}$, buyer i loses the auction with a zero payoff. Hence, buyer i cannot benefit from using multiple identities in any case when $\theta_i < \max\{\theta_{-i},\rho\}$.

Combining the above three cases establishes ex-post buyer identity compatibility for the second-price auction with any reserve price.

To show uniqueness, notice that ex-post buyer identity compatibility implies strategy-proofness (Lemma 1). It is enough to show that the second-price auction is the unique optimally efficient auction that is strategy-proof, and it becomes the unique optimal auction that is strategy-proof when the reserve price ρ^* is applied.

The allocation rules are pinned down by optimality and efficiency respectively, i.e., for all $B \in \mathcal{N}$ and all $\theta_B \in \Theta^{|B|}$,[77]

$$
\begin{aligned}
q_i^{\text{opt}}(\theta_B) &= \mathbf{1}_{\theta_i > \max\{\theta_{-i},\rho^*\}} \\
q_i^{\text{eff}}(\theta_B) &= \mathbf{1}_{\theta_i > \max\{\theta_{-i}\}}.
\end{aligned}
$$

Let $u_i(\theta_i,\theta_{-i})$ be buyer i's equilibrium payoff when the type profile is $\theta_B = (\theta_i,\theta_{-i})$. Strategy-proofness implies that

$$
\begin{aligned}
u_i(\theta_i,\theta_{-i}) &= \max_{\theta_i' \in \Theta} q_i(\theta_i',\theta_{-i})\,\theta_i - t_i(\theta_i',\theta_{-i}) \\
&= q_i(\theta_i,\theta_{-i})\,\theta_i - t_i(\theta_i,\theta_{-i}).
\end{aligned}
$$

By the envelope theorem, we have

$$
u_i(\theta_i,\theta_{-i}) - u_i(0,\theta_{-i}) = \int_0^{\theta_i} q_i(s,\theta_{-i})\,ds.
$$

Then,

$$
t_i(\theta_i,\theta_{-i}) = q_i(\theta_i,\theta_{-i})\,\theta_i - \int_0^{\theta_i} q_i(s,\theta_{-i})\,ds - u_i(0,\theta_{-i}).
$$

Ex-post individual rationality implies that $u_i(0,\theta_{-i}) \geq 0$. To maximize revenue, we have $u_i(0,\theta_{-i}) = 0$. Plugging the optimal and efficient allocation rules into the above equation respectively, we have

$$
\begin{aligned}
t_i^{\text{opt}}(\theta_B) &= \mathbf{1}_{\theta_i > \max\{\theta_{-i},\rho^*\}} \times \max\{\theta_{-i},\rho^*\} \\
t_i^{\text{opt-eff}}(\theta_B) &= \mathbf{1}_{\theta_i > \max\{\theta_{-i}\}} \times \max\{\theta_{-i}\},
\end{aligned}
$$

[77] We can ignore ties in the continuous type space.

which are the second-price auction with and without the reserve price ρ^* respectively. Hence, we have shown uniqueness under strategy-proofness. $\square$

A.2 Proof of Theorem 2

We prove by showing that, for any optimal (or optimally efficient) lit auction, and any set of buyers B, the seller can always strictly benefit from using a single identity. The following argument focuses on optimal auctions, but the same logic applies to optimally efficient auctions by letting the reserve price ρ^* be zero.

Consider a set of bidders $N = B \cup \{0\}$, where bidder 0 is the identity controlled by the seller and every one else is a distinct buyer. Let $\theta_{-i} = \theta_{B\setminus\{i\}}$ and $\overline{\theta}_{-i} = \max\left\{\theta_{B\setminus\{i\}}\right\}$. For any buyer $i \in B$, we have for all $\theta_B \in \Theta^{|B|}$, and all $\theta_0 \in \Theta$,[78]

$$t_i^{\text{opt}}\left(\theta_B, \theta_0\right) \leq \sup_{\theta_0' \in \Theta} t_i^{\text{opt}}\left(\theta_B, \theta_0'\right).$$

In particular, conditional on buyer i of type $\theta_i \geq \rho^*$ winning the auction,[79] we have

$$\begin{aligned} &\mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0\right)\middle| \max\left\{\overline{\theta}_{-i}, \theta_0\right\} < \theta_i\right] \\ \leq &\mathbb{E}_{\theta_{-i},\theta_0}\left[\sup_{\theta_0' \in \Theta} t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0'\right)\middle| \max\left\{\overline{\theta}_{-i}, \theta_0\right\} < \theta_i\right] \\ = &\mathbb{E}_{\theta_{-i}}\left[\sup_{\theta_0' \in \Theta} t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0'\right)\middle| \overline{\theta}_{-i} < \theta_i\right]. \end{aligned} \tag{1}$$

Notice that (1) is the expected payment for buyer i of type θ_i conditional on winning. The payoff equivalence lemma (Myerson, 1981) shows that the expected payment for buyer i of type θ_i (without conditional on winning) is pinned down by optimality given (interim) individual rationality.[80] Under ex-post individual rationality, (1) is also pinned down by optimality, because the losing payment is always zero. The equality is obtained when the payment $t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0\right)$ does not vary with θ_0 conditional on $\theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}$. Notice that the payment rule treats bidders symmetrically under anonymity. Then, $t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0\right)$ also does not vary with θ_{-i} conditional on $\theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}$. Therefore, the payment for buyer i of type θ_i is fixed conditional on winning, which is exactly the first-price auction

[78] θ_0 captures buyers' view that bidder 0 is a genuine buyer, and θ_0' is the type that the seller pretends to be under this view.

[79] We can ignore ties in the continuous type space. In Online Appendix B.4, we deal with ties explicitly in the finite type space.

[80] The term "interim" implies that buyers know their own types, but only have expectations over others' types.

(with reserve). Consequently, we have

$$\mathbb{E}_{\theta_{-i}}\left[\sup_{\theta_0'\in\Theta} t_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0'\right)\middle|\overline{\theta}_{-i}<\theta_i\right]\geq\beta^n\left(\theta_i\right)=\theta_i-\frac{\int_{\rho^*}^{\theta_i}F^{n-1}\left(x\right)dx}{F^{n-1}\left(\theta_i\right)},$$

where $\beta^n\left(\cdot\right)$ is the equilibrium bidding function in the first-price auction with the reserve price ρ^* when the number of bidders is $n=|N|$.[81] Without loss, we assume $n\geq 2$, i.e., there is at least one buyer and one bidder controlled by the seller in the auction.[82]

The first-price auction (with reserve) proves to be the most resilient against the seller's manipulation through the use of fake identities. Moreover, it provides a lower bound for the expected revenue the seller can obtain by shill bidding. Notice that we can focus on the payment from the buyer of the highest type under optimality, because only the highest type can win and only winners pay. Then,

$$\begin{aligned}&\mathbb{E}_{\theta_B}\left[\sup_{\theta_0'\in\Theta}\sum_{i\in B}t_i^{\text{opt}}\left(\theta_B,\theta_0'\right)\right]\\=&\mathbb{E}_{\tilde{\theta}_B}\left[\sup_{\theta_0'\in\Theta}t_i^{\text{opt}}\left(\theta_i=\tilde{\theta}^{1:n-1},\theta_{-i}=\left(\tilde{\theta}^{2:n-1},\ldots,\tilde{\theta}^{n-1:n-1}\right),\theta_0'\right)\right],\end{aligned}$$

where $\tilde{\theta}^{k:n-1}$ is the kth order statistic among $\tilde{\theta}_B$.[83] Put differently, $\tilde{\theta}^{k:n-1}$ is the kth highest of $n-1$ independent draws from the type space Θ with the distribution $F\left(\cdot\right)$. The equality holds because the payment rule treats bidders symmetrically under anonymity. Since we only care about the buyer of the highest type, we can rewrite the above equation in the following way:

$$\begin{aligned}&\mathbb{E}_{\tilde{\theta}_B}\left[\sup_{\theta_0'\in\Theta}t_i^{\text{opt}}\left(\theta_i=\tilde{\theta}^{1:n-1},\theta_{-i}=\left(\tilde{\theta}^{2:n-1},\ldots,\tilde{\theta}^{n-1:n-1}\right),\theta_0'\right)\right]\\=&\mathbb{E}_{\tilde{\theta}_B}\left[\mathbb{E}_{\tilde{\theta}_B}\left[\sup_{\theta_0'\in\Theta}t_i^{\text{opt}}\left(\tilde{\theta}^{1:n-1},\tilde{\theta}^{2:n-1},\ldots,\tilde{\theta}^{n-1:n-1},\theta_0'\right)\middle|\tilde{\theta}^{1:n-1}\right]\right]\\=&\mathbb{E}_{\tilde{\theta}_B}\left[\mathbb{E}_{\tilde{\theta}_B,\theta_{-i}}\left[\sup_{\theta_0'\in\Theta}t_i^{\text{opt}}\left(\tilde{\theta}^{1:n-1},\theta_{-i},\theta_0'\right)\middle|\tilde{\theta}^{1:n-1}\geq\overline{\theta}_{-i}\right]\right]\\=&\mathbb{E}_{\tilde{\theta}_B}\left[\mathbb{E}_{\theta_{-i}}\left[\sup_{\theta_0'\in\Theta}t_i^{\text{opt}}\left(\tilde{\theta}^{1:n-1},\theta_{-i},\theta_0'\right)\middle|\tilde{\theta}^{1:n-1}\geq\overline{\theta}_{-i}\right]\right]\\\geq&\mathbb{E}_{\tilde{\theta}_B}\left[\beta^n\left(\tilde{\theta}^{1:n-1}\right)\right].\end{aligned}$$

[81] We assume $\beta^n\left(\theta\right)=0$ when $\theta<\rho^*$ for convenience.

[82] By definition, $n>0$ because $0\in N$. When $N=\{0\}$, there are no buyers and, consequently, no sale in any case.

[83] Note that $|B|=|N|-1=n-1$.

The first equality follows from the law of iterated expectations. The second equality follows from anonymity and the fact that the conditional joint distribution of $\tilde{\theta}^{2:n-1}, \ldots, \tilde{\theta}^{n-1:n-1}$ given $\tilde{\theta}^{1:n-1}$, is the same as the joint distribution of the order statistics obtained from $n-2$ independent draws from the type space Θ with the distribution truncated on the left at $\tilde{\theta}^{1:n-1}$. To see this, we first observe that the joint density function of the order statistics is

$$f^{\text{order}}\left(\tilde{\theta}^{1:n-1}, \tilde{\theta}^{2:n-1}, \ldots, \tilde{\theta}^{n-1:n-1}\right) = (n-1)! \prod_{i=1}^{n-1} f\left(\tilde{\theta}^{i:n-1}\right).$$

The density function of the first order statistic is

$$f^{\text{order}}\left(\tilde{\theta}^{1:n-1}\right) = \left(F^{n-1}\left(\tilde{\theta}^{1:n-1}\right)\right)' = (n-1)\, F^{n-2}\left(\tilde{\theta}^{1:n-1}\right) f\left(\tilde{\theta}^{1:n-1}\right).$$

Then, we have

$$\begin{aligned}
&f^{\text{order}}\left(\tilde{\theta}^{2:n-1}, \ldots, \tilde{\theta}^{n-1:n-1} \,\middle|\, \tilde{\theta}^{1:n-1}\right)\\
=&\frac{f^{\text{order}}\left(\tilde{\theta}^{1:n-1}, \tilde{\theta}^{2:n-1}, \ldots, \tilde{\theta}^{n-1:n-1}\right)}{f^{\text{order}}\left(\tilde{\theta}^{1:n-1}\right)}\\
=&(n-2)! \prod_{i=2}^{n-1} \frac{f\left(\tilde{\theta}^{i:n-1}\right)}{F\left(\tilde{\theta}^{1:n-1}\right)},
\end{aligned}$$

which is exactly the joint density function of the order statistics obtained from $n-2$ independent draws from the type space Θ with the distribution truncated on the left at $\tilde{\theta}^{1:n-1}$. Therefore, we can replace the random variables $\tilde{\theta}^{2:n-1}, \ldots, \tilde{\theta}^{n-1:n-1}$ conditional on $\tilde{\theta}^{1:n-1}$ with the independent random variables θ_{-i} conditional on $\overline{\theta}_{-i} \leq \tilde{\theta}^{1:n-1}$.

As a recap, we have characterized the lower bound of the expected revenue for the seller by employing a fake identity in the optimal lit auction for any set of buyers B, i.e.,

$$\mathbb{E}_{\theta_B}\left[\sup_{\theta_0 \in \Theta} \sum_{i \in B} t_i^{\text{opt}}\left(\theta_B, \theta_0\right)\right] \geq \mathbb{E}_{\theta_B}\left[\beta^n\left(\theta^{1:n-1}\right)\right].$$

By the Revenue Equivalence Theorem (Myerson, 1981), the expected revenue from any optimal auction with a set of buyers B is the same as the expected revenue from the first-price auction with the reserve price ρ^* with the same set of buyers. Therefore, we have

$$\mathbb{E}_{\theta_B}\left[\sum_{i \in B} t_i^{\text{opt}}\left(\theta_B\right)\right] = \mathbb{E}_{\theta_B}\left[\beta^{n-1}\left(\theta^{1:n-1}\right)\right],$$

where $\beta^{n-1}(\cdot)$ is the equilibrium bidding function in the first-price auction with the reserve price ρ^* when the number of bidders is $n-1 = |B| = |N|-1$.

Notice that for all $\theta > \rho^*$,

$$\beta^{n-1}(\theta) = \theta - \frac{\int_{\rho^*}^{\theta} F^{n-2}(x)\,dx}{F^{n-2}(\theta)} < \theta - \frac{\int_{\rho^*}^{\theta} F^{n-1}(x)\,dx}{F^{n-1}(\theta)} = \beta^{n}(\theta).$$

Intuitively, buyers bid higher when there are more bidders in the first-price auction, because the auction is perceived as more competitive with more bidders in the auction.

Hence, we have

$$\mathbb{E}_{\theta_B}\left[\sum_{i\in B} t_i^{\text{opt}}(\theta_B)\right] < \mathbb{E}_{\theta_B}\left[\sup_{\theta_0\in\Theta}\sum_{i\in B} t_i^{\text{opt}}(\theta_B,\theta_0)\right].$$

Because B is any arbitrary set of buyers, we have

$$\mathbb{E}_{B}\left[\mathbb{E}_{\theta_B}\left[\sum_{i\in B} t_i^{\text{opt}}(\theta_B)\right]\right] < \mathbb{E}_{B}\left[\mathbb{E}_{\theta_B}\left[\sup_{\theta_0\in\Theta}\sum_{i\in B} t_i^{\text{opt}}(\theta_B,\theta_0)\right]\right],$$

which violates ex-post seller identity compatibility. □

A.3 Proof of Theorem 3

Consider the case $|B| = 1$. Incentive compatibility implies that for all $\theta_i, \theta_i' \in \Theta$,

$$\begin{aligned} q_i(\theta_i)\,\theta_i - t_i(\theta_i) &\geq q_i(\theta_i')\,\theta_i - t_i(\theta_i') \\ q_i(\theta_i)\,\theta_i' - t_i(\theta_i) &\leq q_i(\theta_i')\,\theta_i' - t_i(\theta_i'). \end{aligned} \tag{2}$$

Take the difference of the above two inequalities, we have

$$\left(q_i(\theta_i) - q_i(\theta_i')\right) \times \left(\theta_i - \theta_i'\right) \geq 0.$$

Therefore, the allocation rule $q_i(\theta_i)$ is increasing in θ_i. Consequently, the payment rule $t_i(\theta_i)$ is also increasing in θ_i, i.e., $\left(t_i(\theta_i) - t_i(\theta_i')\right) \times \left(\theta_i - \theta_i'\right) \geq 0$.

When the lit auction (q,t) is ex-post auctioneer identity-compatible, we have for all $\theta_i \in \Theta$, and all $n \in \mathbb{N}$,

$$t_i(\theta_i) \geq \sup_{\substack{\theta_{-i}\in\Theta^{n-1} \\ q_i(\theta_i,\theta_{-i})>0}} \frac{t_i(\theta_i,\theta_{-i})}{q_i(\theta_i,\theta_{-i})}. \tag{3}$$

When the number of buyers is n, the expected payment from buyer i of type θ_i is given as follows

$$\mathbb{E}_{\theta_{-i}}\left[t_i\left(\theta_i,\theta_{-i}\right)\right]=\mathbb{E}_{\theta_{-i}}\left[t_i\left(\theta_i,\theta_{-i}\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})>0}+t_i\left(\theta_i,\theta_{-i}\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})=0}\right].$$

Ex-post individual rationality implies that

$$\mathbb{E}_{\theta_{-i}}\left[t_i\left(\theta_i,\theta_{-i}\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})=0}\right]\leq 0.$$

From the inequality (3), we know that

$$\mathbb{E}_{\theta_{-i}}\left[t_i\left(\theta_i,\theta_{-i}\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})>0}\right]\leq\mathbb{E}_{\theta_{-i}}\left[q_i\left(\theta_i,\theta_{-i}\right)t_i\left(\theta_i\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})>0}\right].$$

Let π^n denote the expected revenue for the seller when the number of buyers is n.

$$\begin{aligned}\pi^n=&\sum_{i=1}^{n}\mathbb{E}_{\theta_i,\theta_{-i}}\left[t_i\left(\theta_i,\theta_{-i}\right)\right]\\=&\sum_{i=1}^{n}\mathbb{E}_{\theta_i}\left[\mathbb{E}_{\theta_{-i}}\left[t_i\left(\theta_i,\theta_{-i}\right)\right]\right]\\\leq&\sum_{i=1}^{n}\mathbb{E}_{\theta_i}\left[\mathbb{E}_{\theta_{-i}}\left[q_i\left(\theta_i,\theta_{-i}\right)t_i\left(\theta_i\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})>0}\right]\right]\\=&\sum_{i=1}^{n}\mathbb{E}_{\theta_i,\theta_{-i}}\left[q_i\left(\theta_i,\theta_{-i}\right)t_i\left(\theta_i\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})>0}\right]\\=&\mathbb{E}_{\theta_i,\theta_{-i}}\left[\sum_{i=1}^{n}q_i\left(\theta_i,\theta_{-i}\right)t_i\left(\theta_i\right)\mathbf{1}_{q_i(\theta_i,\theta_{-i})>0}\right]\end{aligned}$$

Recall that $t_i\left(\theta_i\right)$ is increasing in θ_i and $\sum_{i=1}^{n}q_i\left(\theta_i,\theta_{-i}\right)\leq 1$. Then we have

$$\sum_{i=1}^{n}q_i\left(\theta_i,\theta_{-i}\right)t_i\left(\theta_i\right)\leq t_i\left(\max\left\{\theta_i,\theta_{-i}\right\}\right)$$

and

$$\pi^n\leq\mathbb{E}_{\theta_i,\theta_{-i}}\left[t_i\left(\max\left\{\theta_i,\theta_{-i}\right\}\right)\right].\tag{4}$$

By taking $n=1$ for the inequality (3), we have that conditional on $q_i\left(\theta_i\right)>0$,

$$t_i\left(\theta_i\right)\geq\frac{t_i\left(\theta_i\right)}{q_i\left(\theta_i\right)}.$$

Then when $t_i(\theta_i) > 0$, we have $q_i(\theta_i) = 1$. Let[84]

$$\theta^l = \inf \{\theta_i \in \Theta | \, t_i(\theta_i) > 0\}.$$

Then $t_i(\theta_i) > 0$ and $q_i(\theta_i) = 1$ for all $\theta_i > \theta^l$ because of monotonicity. Plugging $q_i(\theta_i) = q_i(\theta_i') = 1$ into the inequality (2), we have $t_i(\theta_i) = t_i(\theta_i')$ for all $\theta_i, \theta_i' > \theta^l$. Ex-post individual rationality implies that $t_i(\theta_i) \leq \theta_i'$ for all $\theta_i, \theta_i' > \theta^l$. Then, $t_i(\theta_i) \leq \theta^l$ for all $\theta_i > \theta^l$. Hence, given θ^l, the seller maximizes expected revenue by running the posted-price mechanism with the price θ^l, because the inequality (4) is obtained with equality in this case. The only remaining problem for the seller is to determine the optimal posted price θ^l, which always exists because the choice set $\Theta \ni \theta^l$ is compact.[85] In total, we show that the posted-price mechanism with a suitable price generates the highest expected revenue among all lit auctions that are ex-post auctioneer identity-compatible. □

A.4 Proof of Lemma 2

Given the dark auction $\left(\Gamma^{\mathbb{N}}, \Sigma^d, S^d\right)$, we define the direct dark auction by, for all $n \in \mathbb{N}$, and all $N \in \mathcal{N}$ with $|N| = n$,

$$\left(q^d(\theta_N), t^d(\theta_N)\right) = x^{(\Gamma^n, S^n)}(\theta_N) = g^{\Gamma^n}(S^n(\theta_N)) \quad \forall \theta_N \in \Theta^n.$$

Thus, this direct dark auction yields the same mapping from type profiles to allocations and payments as the given one. It remains to show that truth-telling is dark Bayesian incentive-compatible.

For the dark auction $\left(\Gamma^{\mathbb{N}}, \Sigma^d, S^d\right)$, dark Bayesian incentive compatibility (Definition 15) implies that, for all $i \in \mathbb{N}$, all $\theta_i \in \Theta$, and all $\sigma^d = (\sigma^n)_{n\in\mathbb{N}} \in \Sigma^d$,

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} p_i(n)\, \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[u_i\left(g^{\Gamma^n}(S^n(\theta_i), S^n(\theta_{-i})), \theta_i\right)\right] \\
\geq &\sum_{n\in\mathbb{N}} p_i(n)\, \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[u_i\left(g^{\Gamma^n}(\sigma^n, S^n(\theta_{-i})), \theta_i\right)\right],
\end{aligned}$$

where $p_i(n)$ is agent i's belief that the number of agents is n. In particular, $S^d(\theta_i') \in \Sigma^d$ for

[84] When $t_i(\theta_i) \leq 0$ for all $\theta_i \in \Theta$, we know from the inequality (4) that $\pi^n \leq 0$ for all $n \in \mathbb{N}$. Then the auction never generates positive revenue. We can safely ignore those auctions.

[85] The exact optimal posted price depends on both the type distribution and the distribution of the set of buyers. The type space Θ is compact for both continuous and finite type spaces.

all $\theta_i' \in \Theta$. Then, we have

$$\sum_{n\in\mathbb{N}} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(S^n(\theta_i), S^n(\theta_{-i})\right), \theta_i\right)\right]$$
$$\geq \sum_{n\in\mathbb{N}} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(S^n(\theta_i'), S^n(\theta_{-i})\right), \theta_i\right)\right],$$

for all $i \in \mathbb{N}$, and all $\theta_i, \theta_i' \in \Theta$. These inequalities translate into

$$\sum_{n\in\mathbb{N}} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[\theta_i q_i^d(\theta_i, \theta_{-i}) - t_i^d(\theta_i, \theta_{-i})\right]$$
$$\geq \sum_{n\in\mathbb{N}} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[\theta_i q_i^d(\theta_i', \theta_{-i}) - t_i^d(\theta_i', \theta_{-i})\right],$$

for all $i \in \mathbb{N}$, and all $\theta_i, \theta_i' \in \Theta_i$. Hence, truth-telling is dark Bayesian incentive-compatible.

$\square$

A.5 Proof of Proposition 3

We first sum the expected revenue over all sets of buyers $B \in \mathcal{N}$, and then transform it into a summation over all potential buyers $i \in \mathbb{N}$. Changing the order of summation is valid, because the sum is always finite. Recall that $p_i(n) = \mathbb{P}(|\mathcal{B}| = n|\, i \in \mathcal{B})$.

$$\begin{aligned}
\pi^d &= \sum_{B\in\mathcal{N}} \mathbb{P}(\mathcal{B} = B) \sum_{i\in B} \mathbb{E}_{\theta_B\in\Theta^{|B|}}\left[t_i^d(\theta_B)\right] \\
&= \sum_{B\in\mathcal{N}} \sum_{i\in B} \mathbb{P}(\mathcal{B} = B)\,\mathbb{E}_{\theta_i\in\Theta}\left[T_i^{|B|}(\theta_i)\right] \\
&= \sum_{i\in\mathbb{N}} \sum_{\substack{B\in\mathcal{N} \\ \text{s.t. } i\in B}} \mathbb{P}(\mathcal{B} = B)\,\mathbb{E}_{\theta_i\in\Theta}\left[T_i^{|B|}(\theta_i)\right] \\
&= \sum_{i\in\mathbb{N}} \sum_{B\in\mathcal{N}} \mathbb{P}(\mathcal{B} = B, i \in \mathcal{B})\,\mathbb{E}_{\theta_i\in\Theta}\left[T_i^{|B|}(\theta_i)\right] \\
&= \sum_{i\in\mathbb{N}} \sum_{n=1}^{\infty} \mathbb{P}(|\mathcal{B}| = n, i \in \mathcal{B})\,\mathbb{E}_{\theta_i\in\Theta}\left[T_i^n(\theta_i)\right] \\
&= \sum_{i\in\mathbb{N}} \sum_{n=1}^{\infty} p_i(n)\,\mathbb{P}(i \in \mathcal{B})\,\mathbb{E}_{\theta_i\in\Theta}\left[T_i^n(\theta_i)\right] \\
&= \sum_{i\in\mathbb{N}} \mathbb{P}(i \in \mathcal{B})\,\mathbb{E}_{\theta_i\in\Theta}\left[\sum_{n=1}^{\infty} p_i(n)\,T_i^n(\theta_i)\right] \\
&= \sum_{i\in\mathbb{N}} \mathbb{P}(i \in \mathcal{B})\,\mathbb{E}_{\theta_i\in\Theta}\left[T_i^d(\theta_i)\right]
\end{aligned}$$

The expression is intuitive: we still sum the expected payments from all potential buyers $i \in \mathbb{N}$, but now weighted by the probability that they belong to the realized set of buyers. By Lemma 3,

$$\begin{aligned}
\pi^d &= \sum_{i \in \mathbb{N}} \mathbb{P}\left(i \in \mathcal{B}\right) \mathbb{E}_{\theta_i \in \Theta}\left[T_i^d\left(\theta_i\right)\right] \\
&= \sum_{i \in \mathbb{N}} \mathbb{P}\left(i \in \mathcal{B}\right) \left(T_i^d\left(0\right) + \mathbb{E}_{\theta_i \in \Theta}\left[Q_i^d\left(\theta_i\right) v\left(\theta_i\right)\right]\right) \\
&= \sum_{i \in \mathbb{N}} \mathbb{P}\left(i \in \mathcal{B}\right) T_i^d\left(0\right) + \sum_{i \in \mathbb{N}} \mathbb{P}\left(i \in \mathcal{B}\right) \mathbb{E}_{\theta_i \in \Theta}\left[Q_i^d\left(\theta_i\right) v\left(\theta_i\right)\right],
\end{aligned}$$

where,

$$\begin{aligned}
&\sum_{i \in \mathbb{N}} \mathbb{P}\left(i \in \mathcal{B}\right) \mathbb{E}_{\theta_i \in \Theta}\left[Q_i^d\left(\theta_i\right) v\left(\theta_i\right)\right] \\
=&\sum_{i \in \mathbb{N}} \mathbb{P}\left(i \in \mathcal{B}\right) \mathbb{E}_{\theta_i \in \Theta}\left[\sum_{n=1}^{\infty} p_i\left(n\right) \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}}\left[q_i^d\left(\theta_i, \theta_{-i}\right)\right] v\left(\theta_i\right)\right] \\
=&\sum_{i \in \mathbb{N}} \mathbb{E}_{\theta_i \in \Theta}\left[\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\mathcal{B}\right| = n, i \in \mathcal{B}\right) \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}}\left[q_i^d\left(\theta_i, \theta_{-i}\right)\right] v\left(\theta_i\right)\right] \\
=&\sum_{i \in \mathbb{N}} \sum_{n=1}^{\infty} \mathbb{P}\left(\left|\mathcal{B}\right| = n, i \in \mathcal{B}\right) \mathbb{E}_{\theta_B \in \Theta^n}\left[q_i^d\left(\theta_B\right) v\left(\theta_i\right)\right] \\
=&\sum_{i \in \mathbb{N}} \sum_{B \in \mathcal{N}} \mathbb{P}\left(\mathcal{B} = B, i \in \mathcal{B}\right) \mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[q_i^d\left(\theta_B\right) v\left(\theta_i\right)\right] \\
=&\sum_{i \in \mathbb{N}} \sum_{\substack{B \in \mathcal{N} \\ \text{s.t. } i \in B}} \mathbb{P}\left(\mathcal{B} = B\right) \mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[q_i^d\left(\theta_B\right) v\left(\theta_i\right)\right] \\
=&\sum_{B \in \mathcal{N}} \mathbb{P}\left(\mathcal{B} = B\right) \sum_{i \in B} \mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[q_i^d\left(\theta_B\right) v\left(\theta_i\right)\right] \\
=&\sum_{B \in \mathcal{N}} \mathbb{P}\left(\mathcal{B} = B\right) \mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sum_{i \in B} q_i^d\left(\theta_B\right) v\left(\theta_i\right)\right].
\end{aligned}$$

Hence,

$$\pi^d = \sum_{i \in \mathbb{N}} \mathbb{P}\left(i \in \mathcal{B}\right) T_i^d\left(0\right) + \sum_{B \in \mathcal{N}} \mathbb{P}\left(\mathcal{B} = B\right) \mathbb{E}_{\theta_B \in \Theta^{|B|}}\left[\sum_{i \in B} q_i^d\left(\theta_B\right) v\left(\theta_i\right)\right].$$

□

A.6 Proof of Theorem 4

Given the previous discussion, we know that the dark first-price auction with the reserve price ρ^* is both ex-post auctioneer identity-compatible and Bayesian buyer identity-compatible. Now we show the uniqueness. Consider an optimal dark auction $\left(q^d, t^d\right)$. By Lemma 3, optimality implies that $q_i^d(\theta_B) = \mathbf{1}_{\theta_i > \max\{\theta_{-i}, \rho^*\}}$ for all $B \in \mathcal{N}$ and all $\theta_B \in \Theta^{|B|}$.[86] Ex-post auctioneer identity compatibility implies that for all $B \in \mathcal{N}$, conditional on buyer $i \in B$ being the winner, the seller cannot push up the payment via identities. In other words, for all $B \in \mathcal{N}$, $\theta_i \in \Theta$ and all $|S| \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[t_i^d(\theta_i, \theta_{-i})\middle|\max\{\theta_{-i}, \rho^*\} < \theta_i\right]\\ \geq&\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\sup_{\overline{\theta}_S<\theta_i} t_i^d(\theta_i, \theta_{-i}, \theta_S)\middle|\max\{\theta_{-i}, \rho^*\} < \theta_i\right],\end{aligned}$$

where $\theta_S \in \Theta^{|S|}$ and $\overline{\theta}_S = \max\{\theta_S\}$. Ex-post individual rationality implies that $t_i(\theta_B) = 0$ when $\theta_i < \max\{\theta_{-i}, \rho^*\}$. Then, we have

$$\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[t_i^d(\theta_i, \theta_{-i})\right] \geq \mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\sup_{\overline{\theta}_S<\theta_i} t_i^d(\theta_i, \theta_{-i}, \theta_S)\right]. \tag{5}$$

Now, suppose that instead of the seller, buyer i employs the set of identities S with a type profile θ_S such that $\overline{\theta}_S < \theta_i$. Bayesian buyer identity compatibility implies that

$$\begin{aligned}&\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\mathbf{1}_{\theta_i>\max\{\theta_{-i},\rho^*\}} \times \left(\theta_i - t_i^d(\theta_i, \theta_{-i})\right)\right]\middle| i \in B\right]\\ \geq&\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\mathbf{1}_{\theta_i>\max\{\theta_{-i},\rho^*\}} \times \left(\theta_i - t_i^d(\theta_i, \theta_{-i}, \theta_S)\right)\right]\middle| i \in B\right].\end{aligned}$$

Then, for all $\theta_S \in \Theta^{|S|}$ such that $\overline{\theta}_S < \theta_i$, we have

$$\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[t_i^d(\theta_i, \theta_{-i})\right]\middle| i \in B\right] \leq \mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[t_i^d(\theta_i, \theta_{-i}, \theta_S)\right]\middle| i \in B\right]. \tag{6}$$

Combining (5) and (6), we have, for all $\theta_i \in \Theta$, all $|S| \in \mathbb{N}$, and all $\theta_S \in \Theta^{|S|}$ such that $\overline{\theta}_S < \theta_i$,

$$\begin{aligned}\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[t_i^d(\theta_i, \theta_{-i})\right] &= \mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\sup_{\overline{\theta}'_S<\theta_i} t_i^d(\theta_i, \theta_{-i}, \theta'_S)\right]\\ \mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[t_i^d(\theta_i, \theta_{-i})\right] &= \mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[t_i^d(\theta_i, \theta_{-i}, \theta_S)\right].\end{aligned}$$

[86] We can ignore ties, because the probability of a tie is zero in the continuous type space.

Hence, the payment $t_i^d(\theta_i, \theta_{-i}, \theta_S)$ is independent of θ_S conditional on $\overline{\theta}_S < \theta_i$. Notice that the payment rule is anonymous, i.e., it treats each bidder symmetrically. It follows that the payment $t_i^d(\theta_i, \theta_{-i}, \theta_S)$ is independent of θ_{-i} conditional on $\overline{\theta}_{-i} < \theta_i$. Hence, conditional on i being the winner, the winning payment is fixed, i.e., conditional on $\theta_i > \rho^*$, for all $\theta_{-i} \in \Theta^{|B|-1}$ such that $\overline{\theta}_{-i} < \theta_i$, all $|S| \in \mathbb{N}$, and all $\theta_S \in \Theta^{|S|}$ such that $\overline{\theta}_S < \theta_i$,

$$t_i^d(\theta_i, \theta_{-i}, \theta_S) = \mathbb{E}_{\theta_{-i} \in \Theta^{|B|-1}} \left[t_i^d(\theta_i, \theta_{-i}) \middle| \max\{\theta_{-i}, \rho^*\} < \theta_i \right].$$

When $|B| = 1$, the above equality implies that conditional on $\max\{\theta_S, \rho^*\} < \theta_i$, we have

$$t_i^d(\theta_i, \theta_S) = t_i^d(\theta_i) \ \ \forall |S| \in \mathbb{N}.$$

In other words, the winning payment is independent of how many bidders in the auction and what losing bidders' types are.[87] Optimality implies that $t_i^d(\theta_i) = \beta^d(\theta_i)$ when $\theta_i > \rho^*$, leading to

$$t_i^d(\theta_B) = \mathbf{1}_{\theta_i > \max\{\theta_{-i}, \rho^*\}} \times \beta^d(\theta_i) \ \ \forall B \in \mathcal{N}, \ \forall \theta_B \in \Theta^{|B|},$$

which corresponds to the dark first-price auction with the reserve price ρ^*.

It is straightforward to verify that the dark first-price auction (without reserve) is optimally efficient (Proposition 3). Moreover, it is both ex-post auctioneer identity-compatible and Bayesian buyer identity-compatible. Its uniqueness follows directly from the above argument by letting the reserve price be zero. □

A.7 Proof of Theorem 5

Consider any partitional mechanism $\left\{\Gamma^{\mathbb{N}}, (\Sigma^y, S^y)_{y \in Y}, \phi\right\}$. Partitional Bayesian incentive compatibility (Definition 19) implies that for all $i \in \mathbb{N}$, all $\theta_i \in \Theta$, all $y \in Y$, and all $\sigma^y = (\sigma^n)_{n \in \phi^{-1}(y)} \in \Sigma^y$,

$$\begin{aligned} &\sum_{n \in \phi^{-1}(y)} p_i(n|y) \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}} \left[u_i \left(g^{\Gamma^n}(S^n(\theta_i), S^n(\theta_{-i})), \theta_i \right) \right] \\ \geq &\sum_{n \in \phi^{-1}(y)} p_i(n|y) \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}} \left[u_i \left(g^{\Gamma^n}(\sigma^n, S^n(\theta_{-i})), \theta_i \right) \right], \end{aligned} \tag{7}$$

[87] Here we use the assumption that $\mathbb{P}(|\mathcal{B}| = 1) > 0$. Otherwise, we cannot pin down the auction format when participants are sure that there is no shill bidding. For example, if $\mathbb{P}(|\mathcal{B}| = 1) = 0$ and $\mathbb{P}(|\mathcal{B}| = 2) > 0$, then buyers are sure that there is no shill bidding when there are two bidders in the auction. Therefore, we do not need to run a first-price auction when there are two bidders in the auction. Still, when there are more than two bidders in the auction, the above equality pins down a fixed payment which only varies with the winner's own type. In other words, we do require a dark first-price auction when the number of bidders is more than two.

where $p_i(n|\, y)$ is agent i's belief that the number of agents is n under signal y.

Let $p_i(y)$ denote the probability of agent i observing signal y. Then,

$$p_i(y) = \sum_{n\in\phi^{-1}(y)} \mathbb{P}\left(|\mathcal{B}| = n|\, i \in \mathcal{B}\right) = \sum_{n\in\phi^{-1}(y)} p_i(n).$$

Hence, $p_i(n|\, y) = \frac{p_i(n)}{p_i(y)}$ for $y = \phi(n)$ and 0 otherwise. By summing (7) over all signals weighted by the probability of agent i observing each signal, we have

$$\begin{aligned}
&\sum_{y\in Y}\left\{p_i(y)\sum_{n\in\phi^{-1}(y)} p_i(n|\, y)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(S^n(\theta_i), S^n(\theta_{-i})\right), \theta_i\right)\right]\right\}\\
=&\sum_{y\in Y}\left\{\sum_{n\in\phi^{-1}(y)} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(S^n(\theta_i), S^n(\theta_{-i})\right), \theta_i\right)\right]\right\}\\
=&\sum_{n\in\mathbb{N}} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(S^n(\theta_i), S^n(\theta_{-i})\right), \theta_i\right)\right]\\
\geq&\sum_{y\in Y}\left\{p_i(y)\sum_{n\in\phi^{-1}(y)} p_i(n|\, y)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(\sigma^n, S^n(\theta_{-i})\right), \theta_i\right)\right]\right\}\\
=&\sum_{y\in Y}\left\{\sum_{n\in\phi^{-1}(y)} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(\sigma^n, S^n(\theta_{-i})\right), \theta_i\right)\right]\right\}\\
=&\sum_{n\in\mathbb{N}} p_i(n)\,\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[u_i\left(g^{\Gamma^n}\left(\sigma^n, S^n(\theta_{-i})\right), \theta_i\right)\right],
\end{aligned}$$

where the second and last equalities follow from the fact that ϕ is partitional.

Let

$$\begin{aligned}
\Sigma^d &= \times_{y\in Y}\Sigma^y\\
S^d &= (S^y)_{y\in Y} = \left((S^n)_{n\in\phi^{-1}(y)}\right)_{y\in Y}.
\end{aligned}$$

Then, the above inequality holds for all $i \in \mathbb{N}$, all $\theta_i \in \Theta$, and all $\sigma^d = \left((\sigma^n)_{n\in\phi^{-1}(y)}\right)_{y\in Y} = (\sigma^y)_{y\in Y} \in \times_{y\in Y}\Sigma^y = \Sigma^d$. Hence, $\left(\Gamma^{\mathbb{N}}, \Sigma^d, S^d\right)$ is dark Bayesian incentive-compatible. By construction, the two mechanisms induce the same outcome rule, and thus they are equivalent. $\square$

B Online Appendix

B.1 Finite Type Space

We assume a finite type space $\Theta = \left\{\theta^1, \theta^2, \ldots, \theta^K\right\}$, where $f_k = \mathbb{P}\left(\theta_i = \theta^k\right)$ for $k \in \{1, 2, \ldots, K\}$. Assume $\theta^1 = 0$, $\theta^{k+1} > \theta^k$, and $K > 1$.

Myerson (1981) characterizes the optimal auction's allocation rule in terms of the virtual valuation in the continuous type space. Similar ideas apply in the finite type space. We can define the virtual valuation of type θ^k as[88]

$$v\left(\theta^k\right) = \theta^k - \left(\theta^{k+1} - \theta^k\right) \frac{1 - \sum_{m=1}^{k} f_m}{f_k}.$$

Lovejoy (2006) and Elkind (2007) characterize the optimal auction in the finite type space, where the optimal reserve price is

$$\rho^* = \min\left\{\theta^k \in \Theta \middle| v\left(\theta^k\right) \geq 0\right\} = \theta^{k^*}.$$

We observe that when $\theta^K = \rho^*$, the second-price auction with the reserve price ρ^* degenerates to the posted-price mechanism with the price ρ^*. To differentiate between these two mechanisms, we assume that $\theta^K > \rho^*$.

All notions of identity compatibility are defined in the same way as in the continuous type space. The main difference is that the probability of a tie is positive in the finite type space, which means that ties cannot be ignored as before. To avoid repetition, we focus directly on how our results extend to the finite type space, organizing the discussion by the auction format to which each result applies.

Second-Price Auction

Our first main result (Theorem 1) characterizes the second-price auction under ex-post buyer identity compatibility, closely mirroring the characterization under strategy-proofness. We now demonstrate that a similar characterization can be achieved in the finite type space—a feature unique to ex-post buyer identity compatibility.

In the continuous type space, strategy-proofness and optimal efficiency (or optimality) pin down the second-price auction (or with the reserve price ρ^*) (Green and Laffont, 1977; Holmström, 1979; Myerson, 1981). However, this result does not directly extend to the finite type space (Harris and Raviv, 1981; Lovejoy, 2006; Jeong and Pycia, 2023), due to the role of ties.

[88] We define $\theta^{K+1} = \theta^K$.

In the finite type space, the standard second-price auction with the reserve price ρ^*—in which the winner pays exactly the second-highest bid—is not optimal. Instead, strategy-proofness and optimality pin down the *tie-corrected* second-price auction with the reserve price ρ^* (Appendix B.6).[89]

Definition 20. The *tie-corrected* second-price auction is the direct mechanism of the second-price auction with a fixed priority order drawn uniformly at random for breaking ties.

In the tie-corrected second-price auction, instead of paying exactly the second-highest bid, the unique winner pays an amount strictly between the second-highest bid (the tying bid) and the lowest unique winning bid (one tick above the tying bid). Hence, the auction is referred to as "tie-corrected." Although this auction (with reserve) is optimal, it is not ex-post buyer identity-compatible. This is because, under the tie-corrected payment rule, buyers strictly prefer winning the tie over being the unique winner. The risk of losing the tie can be virtually eliminated by employing a large number of identities. Similarly, when ties are broken according to a fixed priority order drawn uniformly at random, the second-price auction (with reserve) is optimal but fails to be even Bayesian buyer identity-compatible, because the probability of drawing a high priority can also be increased by using multiple identities.[90] To summarize, we cannot achieve optimality and ex-post buyer identity compatibility at the same time in the finite type space.[91]

Now, let's turn to efficiency. As above, efficiency and strategy-proofness alone cannot pin down the second-price auction in the finite type space, even if we assume buyers of the lowest type obtain zero payoffs. The winner can be charged either above or below the second-highest bid while still preserving strategy-proofness. As demonstrated earlier, this flexibility in the payment rule enables buyers to exploit by using multiple identities. We show that ex-post buyer identity compatibility eliminates this upward flexibility in the payment rule under strategy-proofness, leading to the following result.

Theorem. *The second-price auction maximizes expected revenue among all efficient auctions that are ex-post buyer identity-compatible.*

The proof is relegated to Appendix B.3. It is important to note that the second-price

[89]Anonymity is assumed in this paper. In the continuous type space, the probability of a tie is zero. This auction degenerates to the standard second-price auction.

[90]The second-price auction with a fixed priority order for breaking ties is considered in Akbarpour and Li (2020). In the finite type space, buyers are not indifferent to how ties are broken in the second-price auction. Given a fixed priority order, buyers with higher priorities pay less. Anonymity requires the priority order to be drawn uniformly at random in advance. Hence, buyers can strictly benefit from using multiple identities in order to increase the chance of drawing a high priority. See Appendix B.5 for details.

[91]Consequently, Proposition 4 holds in the finite type space.

auction does not maximize expected revenue among all efficient auctions.[92] The tie-corrected one does, but it fails to be ex-post buyer identity-compatible. This result holds in the continuous type space as well.[93] As previously discussed, the characterization of the second-price auction under strategy-proofness in the continuous type space cannot be directly extended to the finite type space. Therefore, Theorem 6 stands as a novel characterization of the second-price auction that applies to both finite and continuous type spaces.

Similarly, when extending Proposition 1 to the finite type space, we should be careful with the tie-breaking rule to ensure optimality and Bayesian seller identity compatibility at the same time.

Proposition 5. *The second-price auction with the reserve price ρ^*, which breaks ties according to a fixed priority order drawn uniformly at random, is Bayesian seller identity-compatible.*

Proof. In this auction, the best scenario for the seller is when they are drawn the lowest priorities while participating under multiple identities. It ensures that the seller does not risk winning the auction when tying with buyers. This scenario mirrors the one analyzed in Akbarpour and Li (2020), where the auctioneer considers whether to exaggerate other buyers' bids. They show that the auctioneer has no incentive to do so in the English auction with the reserve price ρ^*, which breaks ties according to a fixed priority order. Consequently, the seller also has no incentive to bid above the reserve price ρ^* in the second auction when assigned the lowest priority.[94] Therefore, regardless of the priority order, bidding above the reserve price ρ^* remains unprofitable for the seller. The number of losing bidders also does not affect the winning payment under a fixed priority order. As a result, this auction is Bayesian seller identity-compatible. □

First-Price Auction

In the finite type space, the dark first-price auction is ex-post auctioneer identity-compatible. However, it is no longer Bayesian buyer identity-compatible, because buyers can benefit from using multiple identities to increase their chances of winning ties. Although we cannot extend Theorem 4 to the finite type space, our main insight holds, i.e., dark auctions mute

[92] In the continuous type space, the second-price auction with the reserve price ρ^* is the unique optimal auction that is ex-post buyer identity-compatible. However, in the finite type space, this auction does not maximize expected revenue among all auctions that are ex-post buyer identity-compatible. See Appendix B.6 for details.

[93] It is implied by Theorem 1.

[94] The second-price auction gives the seller even less information than the English auction.

the channel through which the seller heightens the perceived competition by inflating the number of bidders.

For lit auctions, the impossibility result (Theorem 2) remains valid in the finite type space because the conflict between revenue maximization and fake competition exists irrespective of the type space. The proof is relegated to Appendix B.4. In the continuous type space, we identify the first-price auction as the most challenging for the seller to manipulate. In the finite type space, we must handle ties carefully. Specifically, we identify the *tie-corrected* first-price auction as the most challenging for the seller to manipulate *safely*. Safety means that the seller never runs the risk of winning the auction when shill bidding.[95] The tie correction maximizes the seller's loss for ensuring safety. Still, we show that the seller strictly benefits from using a single identity in any optimal lit auction when doing so.

Definition 21. The *tie-corrected* first-price auction works as follows. Bidders simultaneously report their types, and the highest reported type wins. In case of a tie, the winner pays exactly the reported type.[96] Ties are broken uniformly at random. Otherwise, the winner pays a fixed amount $g^n(\theta)$ that depends on the reported type θ and the number of bidders n, similar to the standard first-price auction.[97]

B.2 Distinguishable Agents

When agents are distinguishable, we can drop the assumption of anonymity. Agents can be heterogenous. The game can depend on the specific identities of the agents. In particular, the disclosure policy ϕ is defined as a surjective mapping from the set of sets of agents $\mathcal{N}$ to a set of signals Y, because not only the number of agents but also their identities matter. The definition of partitional disclosure policies still applies here (Definition 18).

We follow the notation in Section 2.1. Consider a collection of games $\Gamma^{\mathcal{N}} = \left\{\Gamma^N\right\}_{N\in\mathcal{N}}$, where each game Γ^N is designed for each finite set of agents $N \in \mathcal{N}$. The set of agents $\mathcal{B}$ is random. Consider any partitional disclosure policy ϕ. When the realized set of agents is

[95] Komo, Kominers and Roughgarden (2024) assumes that the Dutch auction breaks ties deterministically according to an exogenous fixed priority order, providing the seller with prior knowledge of whether a tie favors them. When the number of bidders is random, as in this paper, we must be explicit about how the order is determined if such an order is adopted. Under anonymity, the priority order should be drawn uniformly at random. Theorem 2 holds true irrespective of the tie-breaking rule as long as it satisfies anonymity, because we focus on the safe deviations of the seller. In general, our setting provides the seller with slightly less information, exposing them to the risk of winning the auction in the event of a tie.

[96] In the continuous type space, the probability of a tie is zero. This auction degenerates to the first-price auction.

[97] The fixed amount $g^n(\theta)$ is determined by incentive compatibility constraints. See Appendix B.7 for details.

concealed, agents are allowed to play contingent strategies, i.e., a sequence of strategies that depend on the set of agents. Recall that Σ_i^N denotes the set of strategies for agent $i \in N$ in the game Γ^N. When agent i observes a signal $y \in Y$, the set of strategies for agent i under signal y is denoted by $\Sigma_i^y \subseteq \times_{N \in \phi^{-1}(y)} \Sigma_i^N$, with generic element $\sigma_i^y = \left(\sigma_i^N\right)_{N \in \phi^{-1}(y)}$. As noted earlier, Σ_i^y does not have to be identical to $\times_{N \in \phi^{-1}(y)} \Sigma_i^N$. Under each signal y, a type-strategy $S_i^y = \left(S_i^N\right)_{N \in \phi^{-1}(y)}$ for agent i maps from types to strategies, i.e., $S_i^y : \Theta_i \to \Sigma_i^y$. Let $\Sigma_{\mathbb{N}}^y = (\Sigma_i^y)_{i \in \mathbb{N}}$ and $S_{\mathbb{N}}^y = (S_i^y)_{i \in \mathbb{N}}$ denote the strategy space and strategy profile respectively. We characterize mechanisms under partitional disclosure policies as follows. Note that lit and dark mechanisms are special examples, where $Y^{\text{lit}} = \mathcal{N}, \phi^{\text{lit}} = \text{id}_{\mathcal{N}}$ and $Y^{\text{dark}} = \{d\}, \phi^{\text{dark}} \equiv d$.

Definition 22. $\left\{(\Gamma^y, \Sigma_{\mathbb{N}}^y, S_{\mathbb{N}}^y)_{y \in Y}, \phi\right\}$ is *partitional Bayesian incentive-compatible* if, for all $i \in \mathbb{N}$, all $\theta_i \in \Theta_i$, and all $y \in Y$,

$$S_i^y(\theta_i) \in \arg\max_{\sigma_i^y \in \Sigma_i^y} \sum_{N \in \phi^{-1}(y)} p_i(N|\, y)\, \mathbb{E}_{\theta_{-i} \in \Theta^{n-1}} \left[u_i^N \left(g^{\Gamma^N} \left(\sigma_i^N, S_{-i}^N(\theta_{-i})\right), \theta_i\right)\right],$$

where $p_i(N|\, y) = \mathbb{P}(\mathcal{B} = N|\, i \in \mathcal{B}, y)$ is agent i's belief that the set of agents is N under signal y. In particular, we have:

- $\left(\Gamma^{\mathcal{N}}, S_{\mathbb{N}}^{\mathcal{N}}\right)$ is *lit Bayesian incentive-compatible* if, for all $N \in \mathcal{N}$, all $i \in N$, and all $\theta_i \in \Theta_i$,

$$S_i^N(\theta_i) \in \arg\max_{\sigma_i^N \in \Sigma_i^N} \mathbb{E}_{\theta_{-i} \in \Theta_{N \setminus \{i\}}} \left[u_i^N \left(g^{\Gamma^N} \left(\sigma_i^N, S_{-i}^N(\theta_{-i})\right), \theta_i\right)\right].$$

- $\left(\Gamma^{\mathcal{N}}, \Sigma_{\mathbb{N}}^d, S_{\mathbb{N}}^d\right)$ is *dark Bayesian incentive-compatible* if, for all $i \in \mathbb{N}$ and all $\theta_i \in \Theta_i$,

$$S_i^d(\theta_i) \in \arg\max_{\sigma_i^d \in \Sigma_i^d} \sum_{N \in \mathcal{N}} p_i(N)\, \mathbb{E}_{\theta_{-i} \in \Theta_{N \setminus \{i\}}} \left[u_i^N \left(g^{\Gamma^N} \left(\sigma_i^N, S_{-i}^N(\theta_{-i})\right), \theta_i\right)\right],$$

  where $p_i(N) = \mathbb{P}(\mathcal{B} = N|\, i \in \mathcal{B})$ is agent i's belief that the set of agents is N.

Theorem 5 extends to the case of distinguishable agents. The proof follows Appendix A.7 but requires a slight change of notation. We omit the details for brevity.

B.3 Proof of Theorem 6

As shown in Appendix A.1, the second-price auction is efficient and ex-post buyer identity compatible. We only need to show the remaining part that it maximizes expected revenue among all such auctions.

Anonymity entails the symmetric tie-breaking rule. Then, efficiency implies that

$$q_i^{\mathrm{eff}}(\theta_B) = \frac{1}{|W^{\mathrm{eff}}(\theta_B)|}, \text{ where } W^{\mathrm{eff}}(\theta_B) = \{ i \in B | \, \theta_i = \max\{\theta_B\}\}.$$

By Lemma 1, ex-post buyer identity compatibility implies strategy-proofness. Hence, for each buyer $i \in B$, given any other buyers' type profile $\theta_{-i} \in \Theta^{|B|-1}$, if i is the unique winner by playing the equilibrium strategy of type θ_i, the winning payment must be fixed, i.e., $t_i^{\mathrm{eff}}(\theta_i, \theta_{-i}) = t_i^{\mathrm{w}}(\theta_{-i})$ for all $\theta_i > \max\{\theta_{-i}\}$; if i ties with other buyers by playing the equilibrium strategy of type θ_i, i.e., $\theta_i = \max\{\theta_{-i}\}$, the payment conditional on winning $t_i^{\mathrm{t}}(\theta_{-i})$ must be weakly lower than $t_i^{\mathrm{w}}(\theta_{-i})$ by strategy-proofness, and be weakly lower than $\max\{\theta_{-i}\}$ by ex-post individual rationality. Therefore, we have

$$t_i^{\mathrm{t}}(\theta_{-i}) = \frac{t_i^{\mathrm{eff}}(\max\{\theta_{-i}\}, \theta_{-i})}{q_i^{\mathrm{eff}}(\max\{\theta_{-i}\}, \theta_{-i})} \leq \min\{t_i^{\mathrm{w}}(\theta_{-i}), \max\{\theta_{-i}\}\}.$$

When buyer i has type $\theta_i > \max\{\theta_{-i}\}$, buyer i obtains an equilibrium payoff of $\theta_i - t_i^{\mathrm{w}}(\theta_{-i})$ by using only one identity. With a set of identities N_i, buyer i can play the equilibrium strategy of type $\theta_j = \max\{\theta_{-i}\}$ for all $j \in N_i$, then i wins the tie with probability

$$\frac{|N_i|}{|W^{\mathrm{eff}}(\theta_{-i}, \theta_{N_i})|} > \frac{|N_i|}{|N_i| + |B|}.$$

The payment conditional on winning is $t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i \setminus \{i\}}\right)$.[98] Ex-post buyer identity compatibility ensures that buyer i cannot profit from using any number of identities in any case. In particular, for any $|N_i| \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\left(\theta_i - t_i^{\mathrm{w}}(\theta_{-i})\right)\mathbf{1}_{\theta_i > \max\{\theta_{-i}\}}\right]\middle| \, i \in B\right] \\
&+ \mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\frac{\theta_i - t_i^{\mathrm{t}}(\theta_{-i})}{|W^{\mathrm{eff}}(\theta_B)|}\mathbf{1}_{\theta_i = \max\{\theta_{-i}\}}\right]\middle| \, i \in B\right] \qquad (8)\\
\geq &\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\frac{|N_i|\left(\theta_i - t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right)\right)}{|W^{\mathrm{eff}}(\theta_{-i}, \theta_{N_i})|}\mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right]\middle| \, i \in B\right] \\
> &\mathbb{E}_B\left[\mathbb{E}_{\theta_{-i}\in\Theta^{|B|-1}}\left[\frac{|N_i|\left(\theta_i - t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right)\right)}{|N_i| + |B|}\mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right]\middle| \, i \in B\right] \\
= &\sum_{n\in\mathbb{N}} \mathbb{P}\left(|\mathcal{B}| = n | \, i \in \mathcal{B}\right)\mathbb{E}_{\theta_{-i}\in\Theta^{n-1}}\left[\frac{|N_i|\left(\theta_i - t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right)\right)}{|N_i| + n}\mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right]. \qquad (9)
\end{aligned}$$

[98] By anonymity, we have $t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right) = t_j^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{j\}}\right)$ for all $i, j \in N_i$.

The left-land side (8) is the equilibrium payoff of buyer i when using only one identity, while the right-hand side (9) is the deviation payoff of buyer i when using a set of identities N_i and following the strategy that $\theta_j = \max\{\theta_{-i}\}$ for all $j \in N_i$. Ex-post buyer identity compatibility entails that (9) serves as a lower bound for (8). When an auction is efficient, maximizing revenue is equivalent to minimizing buyers' equilibrium payoffs. Thus, to prove the theorem, it suffices to show that the second-price auction attains the lowest equilibrium payoffs for buyers among all efficient auctions that are ex-post buyer identity-compatible.

Because the expected number of buyers is finite, buyer i's expected number of competitors is also finite,[99] i.e.,

$$\sum_{n\in\mathbb{N}} n\mathbb{P}\left(|\mathcal{B}| = n |\, i \in \mathcal{B}\right) = \frac{\sum_{n\in\mathbb{N}} n\mathbb{P}\left(|\mathcal{B}| = n, i \in \mathcal{B}\right)}{\mathbb{P}\left(i \in \mathcal{B}\right)} \leq \frac{\sum_{n\in\mathbb{N}} n\mathbb{P}\left(|\mathcal{B}| = n\right)}{\mathbb{P}\left(i \in \mathcal{B}\right)} < \infty.$$

Therefore, when buyer i employs an arbitrarily large number of identities, we can interchange limits as follows:

$$\begin{aligned}
&\lim_{|N_i|\to\infty} \sum_{n\in\mathbb{N}} \mathbb{P}\left(|\mathcal{B}| = n |\, i \in \mathcal{B}\right) \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[\frac{|N_i|\left(\theta_i - t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right)\right)}{|N_i| + n} \mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right] \\
&= \sum_{n\in\mathbb{N}} \mathbb{P}\left(|\mathcal{B}| = n |\, i \in \mathcal{B}\right) \lim_{|N_i|\to\infty} \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[\frac{|N_i|\left(\theta_i - t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right)\right)}{|N_i| + n} \mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right] \\
&= \sum_{n\in\mathbb{N}} \mathbb{P}\left(|\mathcal{B}| = n |\, i \in \mathcal{B}\right) \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[\lim_{|N_i|\to\infty} \frac{|N_i|\left(\theta_i - t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right)\right)}{|N_i| + n} \mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right] \\
&= \sum_{n\in\mathbb{N}} \mathbb{P}\left(|\mathcal{B}| = n |\, i \in \mathcal{B}\right) \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[\left(\theta_i - t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right)\right) \mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right] \\
&\geq \sum_{n\in\mathbb{N}} \mathbb{P}\left(|\mathcal{B}| = n |\, i \in \mathcal{B}\right) \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[\left(\theta_i - \max\{\theta_{-i}\}\right) \mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right],
\end{aligned}$$

where the last inequality follows from the fact that

$$t_i^{\mathrm{t}}\left(\theta_{-i}, \theta_{N_i\setminus\{i\}}\right) \leq \max\left\{\theta_{-i}, \theta_{N_i\setminus\{i\}}\right\} = \max\{\theta_{-i}\}.$$

Hence, the equilibrium payoff of buyer i in any efficient auction that is ex-post buyer identity-compatible satisfies:

$$(8) \geq \lim_{|N_i|\to\infty} (9) = \sum_{n\in\mathbb{N}} \mathbb{P}\left(|\mathcal{B}| = n |\, i \in \mathcal{B}\right) \mathbb{E}_{\theta_{-i}\in\Theta^{n-1}} \left[\left(\theta_i - \max\{\theta_{-i}\}\right) \mathbf{1}_{\theta_i \geq \max\{\theta_{-i}\}}\right].$$

[99]We can ignore buyers who participate in the auction with probability zero, i.e., $\mathbb{P}(i \in \mathcal{B}) = 0$.

Since the second-price auction attains this lower bound, it maximizes expected revenue among all efficient auctions that are ex-post buyer identity-compatible. □

B.4 Proof of Theorem 2 (Finite Type Space)

As in Appendix A.2, we prove by showing that, for any optimal (or optimally efficient) lit auction, and any set of buyers B, the seller can always strictly benefit from using a single identity. In the finite type space, we demonstrate that the seller can achieve this through only safe deviations—i.e., deviations in which the seller never wins the auction when shill bidding. Hence, the result holds irrespective of the tie-breaking rule as long as it satisfies anonymity. The following argument focuses on optimal auctions, but the same logic applies to optimally efficient auctions by letting the reserve price ρ^* be zero.

Consider a set of bidders $N = B \cup \{0\}$, where bidder 0 is the identity controlled by the seller and every one else is a distinct buyer. The optimal allocation rule requires that the item is always sold to the bidder of the highest type conditional on being above the reserve price ρ^* (Lovejoy, 2006; Elkind, 2007). Now we focus on the expected payment of buyer $i \in B$ conditional on being the highest type $\theta_i = \max\{\theta_B, \theta_0\} > \rho^*$.[100] Notice that being the highest type is not equivalent to winning the auction because of ties. Let $\theta_{-i} = \theta_{B\setminus\{i\}}$ and $\overline{\theta}_{-i} = \max\left\{\theta_{B\setminus\{i\}}\right\}$. We obtain the following breakdown of this expected payment:

$$
\begin{aligned}
T_i^N(\theta_i) =& \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}(\theta_i, \theta_{-i}, \theta_0) \middle| \theta_i \geq \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right] \\
=& x_1 \times \mathbb{P}_{x_1} + x_2 \times \mathbb{P}_{x_2} + x_3 \times \mathbb{P}_{x_3} + x_4 \times \mathbb{P}_{x_4},
\end{aligned}
$$

where,

$$
\begin{aligned}
x_1 =& \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}(\theta_i, \theta_{-i}, \theta_0) \middle| \theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right], \\
x_2 =& \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}(\theta_i, \theta_{-i}, \theta_0) \middle| \theta_i = \theta_0 > \overline{\theta}_{-i}\right], \\
x_3 =& \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}(\theta_i, \theta_{-i}, \theta_0) \middle| \theta_i = \overline{\theta}_{-i} > \theta_0\right], \\
x_4 =& \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}(\theta_i, \theta_{-i}, \theta_0) \middle| \theta_i = \theta_0 = \overline{\theta}_{-i}\right], \\
\mathbb{P}_{x_1} =& \mathbb{P}\left(\theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\} \middle| \theta_i \geq \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right), \\
\mathbb{P}_{x_2} =& \mathbb{P}\left(\theta_i = \theta_0 > \overline{\theta}_{-i} \middle| \theta_i \geq \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right),
\end{aligned}
$$

[100] Under optimality, buyers of type ρ^* always pay exactly ρ^* conditional on winning. We assume the existence of a type $\theta_i > \rho^*$; otherwise, the highest type would be $\theta^K = \rho^*$, in which case the optimal auction reduces to the posted-price mechanism with the price ρ^*, which is ex-post seller identity-compatible.

$$
\begin{aligned}
\mathbb{P}_{x_3} =& \mathbb{P}\left(\theta_i = \overline{\theta}_{-i} > \theta_0 \middle| \theta_i \geq \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right), \\
\mathbb{P}_{x_4} =& \mathbb{P}\left(\theta_i = \theta_0 = \overline{\theta}_{-i} \middle| \theta_i \geq \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right).
\end{aligned}
$$

Now we discuss how the seller can at least push up the payment of buyer $i \in B$ safely by pretending to be bidder 0 of type θ_0' if the seller knows the type profile of buyers θ_B. Safety for the seller means that the seller will never tie with buyer i and end up winning the auction with some probability.

1. $\theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}$: The seller will maximize pointwise the payment for each type profile. We have

$$
\begin{aligned}
x_1' &= \mathbb{E}_{\theta_{-i},\theta_0}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0'\right) \middle| \theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right] \\
&\geq \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0\right) \middle| \theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right] \\
&= x_1.
\end{aligned}
$$

2. $\theta_i = \theta_0 > \overline{\theta}_{-i}$: The seller will pretend to be a bidder of a lower type $\theta_0' < \theta_i$ to avoid winning the auction with some probability. Moreover, the seller will do it in the optimal way as in the case where $\theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}$. Hence, we have

$$
\begin{aligned}
x_2' &= \mathbb{E}_{\theta_{-i},\theta_0}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0'\right) \middle| \theta_i = \theta_0 > \overline{\theta}_{-i}\right] \\
&= \mathbb{E}_{\theta_{-i},\theta_0}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0'\right) \middle| \theta_i > \max\left\{\overline{\theta}_{-i}, \theta_0\right\}\right] \\
&= x_1' \geq x_1.
\end{aligned}
$$

3. $\theta_i = \overline{\theta}_{-i} > \theta_0$: The seller will maximize pointwise the payment for each type profile. We have

$$
\begin{aligned}
x_3' &= \mathbb{E}_{\theta_{-i},\theta_0}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0'\right) \middle| \theta_i = \overline{\theta}_{-i} > \theta_0\right] \\
&\geq \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0\right) \middle| \theta_i = \overline{\theta}_{-i} > \theta_0\right] \\
&= x_3.
\end{aligned}
$$

4. $\theta_i = \overline{\theta}_{-i} = \theta_0$: The seller will pretend to be a bidder of a lower type $\theta_0' < \theta_i$ to avoid winning the auction with some probability. Moreover, the seller will do it in the

optimal way as in the case where $\theta_i = \overline{\theta}_{-i} > \theta_0$. Hence, we have

$$
\begin{aligned}
x_4' &= \mathbb{E}_{\theta_{-i},\theta_0}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0'\right)\middle|\, \theta_i = \theta_0 = \overline{\theta}_{-i}\right] \\
&= \mathbb{E}_{\theta_{-i},\theta_0}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0'\right)\middle|\, \theta_i = \overline{\theta}_{-i} > \theta_0\right] \\
&= x_3' \geq x_3.
\end{aligned}
$$

In total, by pretending to be bidder 0 of type θ_0', the seller can at least safely push up the expected payment of buyer $i \in B$ conditional on being the highest type $\theta_i = \max\{\theta_B\} > \rho^*$ to

$$
\begin{aligned}
\hat{T}_i^N(\theta_i) &= \mathbb{E}_{\theta_{-i}}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0'\right)\middle|\, \theta_i \geq \overline{\theta}_{-i}\right] \\
&= x_1' \times \mathbb{P}_{x_1} + x_2' \times \mathbb{P}_{x_2} + x_3' \times \mathbb{P}_{x_3} + x_4' \times \mathbb{P}_{x_4} \\
&\geq x_1 \times \mathbb{P}_{x_1} + x_1 \times \mathbb{P}_{x_2} + x_3 \times \mathbb{P}_{x_3} + x_3 \times \mathbb{P}_{x_4} \\
&= x_1 \times \left(\mathbb{P}_{x_1} + \mathbb{P}_{x_2}\right) + x_3 \times \left(\mathbb{P}_{x_3} + \mathbb{P}_{x_4}\right) \\
&= (1+h)\left(x_1 \times \mathbb{P}_{x_1} + x_3 \times \mathbb{P}_{x_3}\right),
\end{aligned}
\tag{10}
$$

where

$$
h = \frac{\mathbb{P}_{x_2}}{\mathbb{P}_{x_1}} = \frac{\mathbb{P}_{x_4}}{\mathbb{P}_{x_3}} = \frac{\mathbb{P}\left(\theta_i = \theta_0 \middle|\, \theta_i \geq \max\left\{\overline{\theta}_{-i},\theta_0\right\}\right)}{\mathbb{P}\left(\theta_i > \theta_0 \middle|\, \theta_i \geq \max\left\{\overline{\theta}_{-i},\theta_0\right\}\right)},
$$

because types are independent. Ex-post individual rationality implies that

$$
\begin{aligned}
x_2 &= \mathbb{E}_{\theta_{-i},\theta_0}\left[t_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0\right)\middle|\, \theta_i = \theta_0 > \overline{\theta}_{-i}\right] \\
&\leq \theta_i \mathbb{E}_{\theta_{-i},\theta_0}\left[q_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0\right)\middle|\, \theta_i = \theta_0 > \overline{\theta}_{-i}\right], \\
x_4 &= \mathbb{E}_{\theta_{-i},\theta_0}\left[q_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0\right)\middle|\, \theta_i = \theta_0 = \overline{\theta}_{-i}\right] \\
&\leq \theta_i \mathbb{E}_{\theta_{-i},\theta_0}\left[q_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0\right)\middle|\, \theta_i = \theta_0 = \overline{\theta}_{-i}\right].
\end{aligned}
$$

Then,

$$
\begin{aligned}
& x_1 \times \mathbb{P}_{x_1} + x_3 \times \mathbb{P}_{x_3} \\
=& T_i^N(\theta_i) - \left(x_2 \times \mathbb{P}_{x_2} + x_4 * \mathbb{P}_{x_4}\right) \\
\geq& T_i^N(\theta_i) - \theta_i \mathbb{E}_{\theta_{-i},\theta_0}\left[q_i^{\text{opt}}\left(\theta_i,\theta_{-i},\theta_0\right)\middle|\, \theta_i = \theta_0 \geq \overline{\theta}_{-i}\right]\mathbb{P}\left(\theta_i = \theta_0 \middle|\, \theta_i \geq \max\left\{\overline{\theta}_{-i},\theta_0\right\}\right).
\end{aligned}
\tag{11}
$$

Notice that optimality under anonymity pins down the expected payment of buyer i.[101] Ex-post individual rationality implies that the payment conditional on not being the highest or $\theta_i < \rho^*$ is zero. Conditional on winning the tie, buyer i of type ρ^* pays exactly ρ^* under optimality. Hence, the expected payment of buyer i conditional on being the highest type $\theta_i > \rho^*$, i.e., $T_i^N(\theta_i)$, is fixed. Combining (10) and (11), we observe that the minimum of $\hat{T}_i^N(\theta_i)$ can be obtained by the *tie-corrected* first-price auction with the reserve price ρ^*: when bidders tie with type $\theta \geq \rho^*$, they pay θ.[102] Ties are broken uniformly at random. Otherwise, the unique winner pays a fixed payment $g^n(\theta) \geq \rho^*$, where $n = |N|$ is the number of bidders in the auction. The exact value of $g^n(\theta)$ is determined by incentive compatibility constraints.[103] (See Appendix B.7.)

Intuitively, in the tie-corrected first-price auction with the reserve price ρ^*, the seller incurs the greatest loss by not tying with buyers. Furthermore, the seller cannot manipulate the payment when they lose the auction. This is the "worst" optimal auction for the seller, which characterizes the lowest expected payment from buyer i to the seller who pretends to be bidder 0 safely, i.e.,

$$\begin{aligned}
&\mathbb{E}_{\theta_{-i}}\left[\sup_{\theta_0'<\theta_i} t_i^{\text{opt}}\left(\theta_i, \theta_{-i}, \theta_0'\right)\right] \\
\geq &\mathbb{E}_{\theta_{-i}}\left[g^n\left(\theta_i\right) \times \mathbf{1}_{\theta_i > \bar{\theta}_{-i}} + \theta_i q_i^{\text{opt}}\left(\theta_i, \theta_{-i}\right) \times \mathbf{1}_{\theta_i = \bar{\theta}_{-i}}\right].
\end{aligned}$$

Notice that conditional on $\theta_i = \theta_j$, the payment for buyer i is the same as that for buyer j under anonymity, i.e., $t_i^{\text{opt}}(\theta_B, \theta_0') = t_j^{\text{opt}}(\theta_B, \theta_0')$. In particular, we have

$$\begin{aligned}
&\sup_{\theta_0'<\theta_i=\theta_j=\bar{\theta}_B} t_i^{\text{opt}}\left(\theta_B, \theta_0'\right) + \sup_{\theta_0'<\theta_i=\theta_j=\bar{\theta}_B} t_j^{\text{opt}}\left(\theta_B, \theta_0'\right) \\
= &\sup_{\theta_0'<\theta_i=\theta_j=\bar{\theta}_B} \left\{t_i^{\text{opt}}\left(\theta_B, \theta_0'\right) + t_j^{\text{opt}}\left(\theta_B, \theta_0'\right)\right\}.
\end{aligned}$$

In general, we have

$$\sup_{\theta_0'<\bar{\theta}_B} \sum_{i\in B} t_i^{\text{opt}}\left(\theta_B, \theta_0'\right) = \sum_{i\in B} \sup_{\theta_0'<\bar{\theta}_B} t_i^{\text{opt}}\left(\theta_B, \theta_0'\right).$$

Then, we can characterize the lower bound of the expected revenue for the seller by employing

[101] In particular, it is pinned down by the local downward incentive compatibility constraints which bind under optimality (Lovejoy, 2006; Elkind, 2007).

[102] In the continuous type space, the probability of a tie is zero. This auction degenerates to the first-price auction (with reserve).

[103] We assume $g^n(\theta) = 0$ when $\theta < \rho^*$ for convenience.

a fake identity and playing safely in the optimal lit auction for any set of buyers B as follows:

$$\mathbb{E}_{\theta_B}\left[\sup_{\theta_0'<\bar{\theta}_B}\sum_{i\in B} t_i^{\text{opt}}\left(\theta_B,\theta_0'\right)\right]=\sum_{i\in B}\mathbb{E}_{\theta_B}\left[\sup_{\theta_0'<\bar{\theta}_B} t_i^{\text{opt}}\left(\theta_B,\theta_0'\right)\right]$$
$$=\sum_{i\in B}\mathbb{E}_{\theta_B}\left[g^n\left(\theta_i\right)\times\mathbf{1}_{\theta_i>\bar{\theta}_{-i}}+\theta_i q_i^{\text{opt}}\left(\theta_i,\theta_{-i}\right)\times\mathbf{1}_{\theta_i=\bar{\theta}_{-i}}\right].$$

By the Revenue Equivalence Theorem in the finite type space (Lovejoy, 2006; Elkind, 2007), any optimal auction for a set of buyers B generates the same expected revenue as in the tie-corrected first-price auction with the reserve price ρ^* for the same set of buyers, since both are optimal auctions. Note that $|B|=|N|-1=n-1$. Then, we have

$$\mathbb{E}_{\theta_B}\left[\sum_{i\in B} t_i^{\text{opt}}\left(\theta_B\right)\right]=\sum_{i\in B}\mathbb{E}_{\theta_B}\left[t_i^{\text{opt}}\left(\theta_B\right)\right]$$
$$=\sum_{i\in B}\mathbb{E}_{\theta_B}\left[g^{n-1}\left(\theta_i\right)\times\mathbf{1}_{\theta_i>\bar{\theta}_{-i}}+\theta_i q_i^{\text{opt}}\left(\theta_i,\theta_{-i}\right)\times\mathbf{1}_{\theta_i=\bar{\theta}_{-i}}\right].$$

In Appendix B.7, we show that for all $\theta_i>\rho^*$, and all $n\in\mathbb{N}$, we have

$$g^{n-1}\left(\theta_i\right)<g^n\left(\theta_i\right).$$

Therefore,

$$\mathbb{E}_{\theta_B}\left[\sum_{i\in B} t_i^{\text{opt}}\left(\theta_B\right)\right]<\mathbb{E}_{\theta_B}\left[\sup_{\theta_0'<\bar{\theta}_B}\sum_{i\in B} t_i^{\text{opt}}\left(\theta_B,\theta_0'\right)\right].$$

Because B is any arbitrary set of buyers, we have

$$\mathbb{E}_B\left[\mathbb{E}_{\theta_B}\left[\sum_{i\in B} t_i^{\text{opt}}\left(\theta_B\right)\right]\right]<\mathbb{E}_B\left[\mathbb{E}_{\theta_B}\left[\sup_{\theta_0<\bar{\theta}_B}\sum_{i\in B} t_i^{\text{opt}}\left(\theta_B,\theta_0\right)\right]\right],$$

which violates ex-post seller identity compatibility.

B.5 Second-Price Auction with a Fixed Priority Order

Assume the fixed priority order is such that we always break ties by giving the item to the lowest-numbered buyer. Then the allocation rule is $q_i^{\text{p-2nd}}\left(\theta_B\right)=\mathbf{1}_{i=W^{\text{p-2nd}}\left(\theta_B\right)}$, where

$$W^{\text{p-2nd}}\left(\theta_B\right)=\min\left\{i\in B\middle|\,\theta_i=\max\left\{\theta_B\right\}\right\},$$

and the payment rule is

$$t_i^{\text{p-2nd}}(\theta_B) = q_i^{\text{p-2nd}}(\theta_B) \times \min\left\{\theta_i' \in \Theta \middle| i = W^{\text{p-2nd}}(\theta_i', \theta_{-i})\right\}.$$

For example, let $B = \{1, 2\}$ and $\Theta = \left\{\theta^L, \theta^H\right\}$, where $\theta^H > \theta^L$. Then[104]

$$t_1^{\text{p-2nd}}\left(\theta^H, \theta^L\right) = \theta^L, \text{ but } t_2^{\text{p-2nd}}\left(\theta^L, \theta^H\right) = \theta^H.$$

Buyer 1 of type θ^H has positive payoff, whereas buyer 2 of type θ^H has zero payoff. In general, the higher the priority, the higher the expected payoff for the buyer in the second-price auction. If the priority order is drawn uniformly at random in advance, buyers can strictly benefit from using multiple identities in order to increase the chance of drawing a high priority. After the priority order is drawn, buyers will bid truthfully for the identity that has the highest priority, and bid the lowest type for all other identities. By doing so, buyers will never be harmed by the additional identities, and the resulting payment rule for buyers is the same as the one by ignoring those identities with lower priorities. Hence, the second-price auction with a fixed priority order drawn uniformly at random for breaking ties is not Bayesian buyer identity-compatible.

Now we consider the tie-corrected second-price auction, which is actually the direct mechanism of the second-price auction with a fixed priority order drawn uniformly at random for breaking ties.

B.6 Tie-Corrected Second-Price Auction

From Myerson (1981), Lovejoy (2006), and Elkind (2007), we know that optimality in the regular auction implies that the seller always gives the item to the buyer who has the highest type conditional on being above the reserve price ρ^*. Under the symmetric tie-breaking rule, which follows from anonymity, the allocation rule is pinned down as $q_i^{\text{opt}}(\theta_B) = \frac{1}{|W^{\text{opt}}(\theta_B)|}$, where

$$W^{\text{opt}}(\theta_B) = \{i \in B | \theta_i = \max\{\theta_B\} \text{ and } \theta_i \geq \rho^*\}.$$

Notice that strategy-proofness is not enough to pin down the payment rule.[105] In the finite type space, Lovejoy (2006) and Elkind (2007) point out that optimal auctions require that the local downward incentive compatibility constraints must be binding, whereas the local upward incentive compatibility constraints can be slack. Specifically, in the optimal auction

[104] $t_i^{\text{p-2nd}}\left(\theta^L, \theta^L\right) = \theta^L$ and $t_i^{\text{p-2nd}}\left(\theta^H, \theta^H\right) = \theta^H$ for all $i \in \{1, 2\}$.

[105] Both the second-price auction and the tie-corrected one are strategy-proof under the same allocation rule.

that is strategy-proof, buyer $i \in B$ of type θ^k finds it indifferent between bidding θ^k and θ^{k-1}, given any other bidders' type profile $\theta_{-i} \in \Theta^{|B|-1}$. Then, the payment rule is pinned down as

$$t_i^{\text{tc-2nd}}(\theta_B) = \begin{cases} \frac{1}{\left|W^{\text{opt}}\left(\theta_i^t, \theta_{-i}\right)\right|} \theta_i^{\text{t}} + \left(1 - \frac{1}{\left|W^{\text{opt}}\left(\theta_i^t, \theta_{-i}\right)\right|}\right) \theta_i^{\text{w}} & \text{if } \theta_i \geq \theta_i^{\text{w}}, \\ \frac{1}{\left|W^{\text{opt}}\left(\theta_i^t, \theta_{-i}\right)\right|} \theta_i^{\text{t}} & \text{if } \theta_i = \theta_i^t, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\theta_i^{\text{w}} = \min \left\{ \theta_i \in \Theta \middle| \, q_i^{\text{opt}}(\theta_i, \theta_{-i}) = 1 \right\}$$

is the lowest unique winning type for buyer i,

$$\theta_i^{\text{t}} = \min \left\{ \theta_i \in \Theta \middle| \, q_i^{\text{opt}}(\theta_i, \theta_{-i}) > 0 \right\}$$

is the tying type for buyer i, and $\frac{1}{\left|W^{\text{opt}}\left(\theta_i^t, \theta_{-i}\right)\right|}$ is the probability of winning for buyer i of the tying type. We refer to this auction as the tie-corrected second-price auction (with reserve), which is pinned down by strategy-proofness and optimality in the finite type space.

For comparison, we also present the second-price auction under the symmetric tie-breaking rule here. The allocation rule is the same as before. The payment rule is

$$t_i^{\text{2nd}}(\theta_B) = \begin{cases} \theta_i^{\text{t}} & \text{if } \theta_i \geq \theta_i^{\text{w}}, \\ \frac{1}{\left|W^{\text{opt}}\left(\theta_i^t, \theta_{-i}\right)\right|} \theta_i^{\text{t}} & \text{if } \theta_i = \theta_i^t, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that buyers pay strictly less in the second-price auction than in the tie-corrected one, where the unique winning bidder i is asked to pay between i's lowest unique winning type (strictly higher than the second-highest bid in the finite type space) and i's tying type (the second-highest bid), with the weight on the tying type equal to the probability that i wins the tie. Hence, the second-price auction under the symmetric tie-breaking rule is not optimal in the finite type space. The two auctions are equivalent when ties occur with probability zero.[106] For example, in the continuous (or some asymmetric) type space, we have $\theta_i^{\text{w}} = \theta_i^{\text{t}}$ and $\left|W^{\text{opt}}(\cdot)\right| \in \{0, 1\}$.

In the previous example, we have $t_1^{\text{tc-2nd}}\left(\theta^H, \theta^L\right) = t_2^{\text{tc-2nd}}\left(\theta^L, \theta^H\right) = \frac{\theta^H + \theta^L}{2}$. Notice that this is also the payment rule when the fixed priority order is drawn uniformly at

[106]If $\mathbb{P}(\theta_i = \max\{\theta_{-i}\}) = 0$, then $\theta_i^{\text{w}} = \theta_i^{\text{t}}$.

random. In fact, the second-price auction with a fixed priority order (with reserve) is also optimal. When the priority order is drawn uniformly at random, the payment rule under randomization is identical to that of the tie-corrected second-price auction. Hence, the tie-corrected second-price auction is the direct mechanism of the second-price auction with a fixed priority order drawn uniformly at random for breaking ties.

We can easily check that the tie-corrected second-price auction is indeed strategy-proof for all buyers $i \in B$. If $\theta_i < \theta_i^t$, buyer i finds it not profitable to win the auction by bidding higher, because the winning payment is strictly higher than θ_i. If $\theta_i = \theta_i^t$, buyer i has payoff zero for truthful bidding, is indifferent to losing the auction, and finds it not profitable to be the unique winner by bidding higher when possible, because the winning payment is strictly higher than θ_i^{t}. If $\theta_i > \theta_i^{\mathrm{w}}$, buyer i find it strictly worse off by bidding θ_i^{t} (or less), because the gain from the decrease in the payment cannot make up the loss from the decrease in the probability of winning. If $\theta_i = \theta_i^{\mathrm{w}}$, buyer i has positive payoff for truthful bidding, finds it strictly worse off by losing the auction, and is indifferent between bidding θ_i^t and θ_i^{w} (or more), because i pays exactly θ_i for the increased probability of winning. Hence, the tie-corrected second-price auction is strategy-proof.

However, the tie-corrected second-price auction is not ex-post buyer identity-compatible. It is always profitable for buyer $i \in B$ of type $\theta_i \geq \theta_i^{\mathrm{w}}$ to bid θ_i^{t} under a sufficiently large number of identities. By doing so, buyer i wins the tie with probability sufficiently close to one, and hence avoids the loss from the decrease in the probability of winning by bidding less. At the same time, the winning payment strictly decreases from the unique winning payment to the tying payment θ^{t}.

The second-price auction is ex-post buyer identity-compatible. However, the second-price auction (with reserve) does not maximize expected revenue among all auctions that are ex-post identity-compatible. For example, if the probability of k buyers showing up in the auction is sufficiently close to one, and the probability of having more than k buyers is sufficiently close to zero, then the auctioneer can run the second-price auction (with reserve) up to $k-1$ bidders, run the tie-corrected one (with reserve) only for k bidders, and commit to no sale for more than k bidders. Then, this auction is ex-post buyer identity-compatible, and generates higher expected revenue than always running the second-price auction, because it performs strictly better when there are k bidders.

B.7 Tie-Corrected First-Price Auction

In the tie-corrected first-price auction with the reserve price ρ^*, bidders report their types in the auction, and whoever reports the highest wins the auction. In case of a tie at the type $\theta^k \geq \rho^*$, bidders pay θ^k by breaking ties uniformly at random. Otherwise, the unique winner pays a fixed amount $g^n\left(\theta^k\right) \geq \rho^*$, where $n = |N|$ is the number of bidders in the auction.[107] Recall that $f_k = \mathbb{P}\left(\theta_i = \theta^k\right)$. The cumulative distribution function is defined as $F_k = \sum_{m=1}^{k} f_m$.[108] Recall that the optimal reserve price is defined as

$$\rho^* = \min\left\{\theta^k \in \Theta \middle| v\left(\theta^k\right) \geq 0\right\} = \theta^{k^*}.$$

The expected payoff of bidder i of type $\theta^k > \theta^{k^*}$ is

$$\begin{aligned} U_i\left(\theta^k \middle| \theta^k\right) &= \left(\theta^k - g^n\left(\theta^k\right)\right)\mathbb{P}\left(\forall j \in N \setminus \{i\} : \theta^k > \theta_j\right) \\ &= \left(\theta^k - g^n\left(\theta^k\right)\right)\mathbb{P}^{n-1}\left(\theta^k > \theta_j\right) \\ &= \left(\theta^k - g^n\left(\theta^k\right)\right)F_{k-1}^{n-1}, \end{aligned}$$

The expected payoff of bidder i when misreporting type θ^{k-1} is

$$\begin{aligned} U_i\left(\theta^{k-1} \middle| \theta^k\right) &= \left(\theta^k - g^n\left(\theta^{k-1}\right)\right)\mathbb{P}\left(\forall j \in N \setminus \{i\} : \theta^{k-1} > \theta_j\right) \\ &\quad + \left(\theta^k - \theta^{k-1}\right)\sum_{m=1}^{n-1}\frac{\mathbb{P}\left(m \text{ additional bidders tie at the type } \theta^{k-1}\right)}{m+1} \\ &= \left(\theta^k - g^n\left(\theta^{k-1}\right)\right)F_{k-2}^{n-1} \\ &\quad + \left(\theta^k - \theta^{k-1}\right)\sum_{m=1}^{n-1}\frac{\binom{n-1}{m}f_{k-1}^m F_{k-2}^{n-1-m}}{m+1}. \end{aligned}$$

Here $\binom{n-1}{m}$ is the number of ways of selecting m bidders out of $n-1$ bidders to tie with bidder i. We have $m+1$ in the denominator because of breaking ties uniformly at random. The summation accounts for all winning probabilities at ties with different numbers of bidders.

$$\begin{aligned} &\sum_{m=1}^{n-1}\frac{\binom{n-1}{m}f_{k-1}^m F_{k-2}^{n-1-m}}{m+1} \\ =&\sum_{m=1}^{n-1}\frac{(n-1)!}{(m+1)!\,(n-1-m)!}f_{k-1}^m F_{k-2}^{n-1-m} \\ =&\sum_{m=2}^{n}\frac{(n-1)!}{m!\,(n-m)!}f_{k-1}^{m-1}F_{k-2}^{n-m} \\ =&\frac{1}{nf_{k-1}}\left(\left(f_{k-1}+F_{k-2}\right)^n - F_{k-2}^n - nf_{k-1}F_{k-2}^{n-1}\right) \end{aligned}$$

[107]We assume that $g^n\left(\theta^k\right) = 0$ when $\theta^k < \rho^*$ for convenience.
[108]$F_k = 0$ when $k < 1$.

$$= \frac{1}{nf_{k-1}} \left(F^n_{k-1} - F^n_{k-2}\right) - F^{n-1}_{k-2}.$$

Then, we have

$$\begin{aligned} U_i\left(\theta^{k-1} \mid \theta^k\right) &= \left(\theta^k - g^n\left(\theta^{k-1}\right)\right) F^{n-1}_{k-2} \\ &\quad + \left(\theta^k - \theta^{k-1}\right)\left[\frac{1}{nf_{k-1}}\left(F^n_{k-1} - F^n_{k-2}\right) - F^{n-1}_{k-2}\right]. \\ &= \left(\theta^{k-1} - g^n\left(\theta^{k-1}\right)\right) F^{n-1}_{k-2} \\ &\quad + \left(\theta^k - \theta^{k-1}\right)\frac{F^n_{k-1} - F^n_{k-2}}{nf_{k-1}}. \end{aligned}$$

Optimality implies that the downward local incentive compatibility constraints must bind in equilibrium (Lovejoy, 2006; Elkind, 2007). Hence, $U_i\left(\theta^k \middle| \theta^k\right) = U_i\left(\theta^{k-1}\middle| \theta^k\right)$, and we have

$$\left(\theta^k - g^n\left(\theta^k\right)\right) F^{n-1}_{k-1} = \left(\theta^{k-1} - g^n\left(\theta^{k-1}\right)\right) F^{n-1}_{k-2} + \left(\theta^k - \theta^{k-1}\right)\frac{F^n_{k-1} - F^n_{k-2}}{nf_{k-1}}.$$

Since bidders of the lowest possible winning type have zero payoffs, it follows that $g^n\left(\theta^{k^*}\right) = \theta^{k^*}$. Therefore,

$$\left(\theta^k - g^n\left(\theta^k\right)\right) F^{n-1}_{k-1} = \sum_{m=k^*+1}^{k}\left(\theta^m - \theta^{m-1}\right)\frac{F^n_{m-1} - F^n_{m-2}}{nf_{m-1}},$$

and

$$g^n\left(\theta^k\right) = \theta^k - \sum_{m=k^*+1}^{k}\left(\theta^m - \theta^{m-1}\right)\frac{F^n_{m-1} - F^n_{m-2}}{nf_{m-1}F^{n-1}_{k-1}} \tag{12}$$

Notice that $g^n\left(\theta^k\right) < \theta^k$ and $\lim_{n\to\infty} g^n\left(\theta^k\right) = \theta^k$.[109] In particular,

$$g^{n+1}\left(\theta^k\right) - g^n\left(\theta^k\right) = \sum_{m=k^*+1}^{k}\left(\theta^m - \theta^{m-1}\right)\left(\frac{F^n_{m-1} - F^n_{m-2}}{nf_{m-1}F^{n-1}_{k-1}} - \frac{F^{n+1}_{m-1} - F^{n+1}_{m-2}}{(n+1)\, f_{m-1}F^n_{k-1}}\right),$$

[109] As a simple check, when $d\theta = \theta^m - \theta^{m-1} \to 0$ and $f_{m-1} \to 0$, the payment rule (12) becomes the equilibrium bidding function in the first-price auction with the reserve price ρ^* in the continuous type space, i.e., $b_i(\theta) = \theta - \frac{\int_{\rho^*}^{\theta} F^{n-1}(s)ds}{F^{n-1}(\theta)}$. Notice that our auction is different from the first-price auction in the finite type space because of the different payment in case of a tie. However, the difference disappears in the continuous type space because the probability of a tie is zero. Hence, it should reduce to the first-price auction with the reserve price ρ^* in the continuous type space when taking limits.

where,

$$
\begin{aligned}
&\frac{F_{m-1}^{n}-F_{m-2}^{n}}{nf_{m-1}F_{k-1}^{n-1}}-\frac{F_{m-1}^{n+1}-F_{m-2}^{n+1}}{\left(n+1\right)f_{m-1}F_{k-1}^{n}} \\
=&\frac{\left(n+1\right)\left(F_{m-1}^{n}-F_{m-2}^{n}\right)F_{k-1}-n\left(F_{m-1}^{n+1}-F_{m-2}^{n+1}\right)}{n\left(n+1\right)f_{m-1}F_{k-1}^{n}} \\
\geq&\frac{\left(n+1\right)\left(F_{m-1}^{n}-F_{m-2}^{n}\right)F_{m-1}-n\left(F_{m-1}^{n+1}-F_{m-2}^{n+1}\right)}{n\left(n+1\right)f_{m-1}F_{k-1}^{n}} \\
=&\frac{\left(F_{m-1}^{n}-F_{m-2}^{n}\right)F_{m-1}+\left[nF_{m-2}-nF_{m-1}\right]F_{m-2}^{n}}{n\left(n+1\right)f_{m-1}F_{k-1}^{n}} \\
=&\frac{\sum_{i=0}^{n-1}\left(F_{m-1}^{i+1}F_{m-2}^{n-1-i}-F_{m-2}^{n}\right)}{n\left(n+1\right)F_{k-1}^{n}} \\
>&\frac{\sum_{i=0}^{n-1}\left(F_{m-2}^{i+1}F_{m-2}^{n-1-i}-F_{m-2}^{n}\right)}{n\left(n+1\right)F_{k-1}^{n}}=0.
\end{aligned}
$$

Hence, for all $\theta^{k}\geq\rho^{*}$,

$$
g^{n}\left(\theta^{k}\right)<g^{n+1}\left(\theta^{k}\right).
$$